755

ODTÜ-2013

PHYS-105

DN

VECTORS

Vector Quantities

Physical quantities that have both numerical and directional quantities.

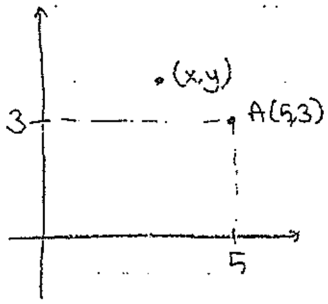
→ addition

→ substraction

Coordinate systems → Cartesion
→ Polar

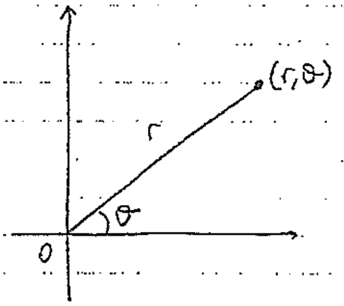
Cartesion: (Rectangular Coordinate System)

Points are labeled (x,y)



Polar:

Origin, reference line are noted. Points are labeled (r,θ)

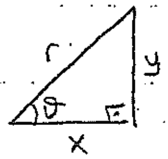


The reference line is often the x-axis.

Polar to Cartesion Coordinate System

$x = r\cos\theta$

$y = r\sin\theta$

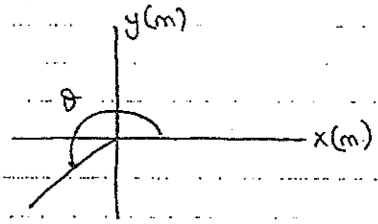


If the cartesion coordinates are known

$\tan\theta = \frac{y}{x}$ $r = \sqrt{x^2+y^2}$

Example:

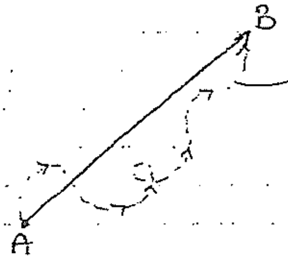
$(x,y) = (-3{,}50,\ -2{,}50)$ m find the polar coordinates.



$\theta = 216°$ $\left(\tan\theta = \frac{y}{x}\right)$

Scalar → no direction

Vectorel → direction, number



→ This is the distance traveled and it's scalar.

The displacement is the solid line from A to B. It's vectorel.
↙
independent of the path taken between A-B.

$\vec{A}$ → vector

$|\vec{A}|$ → magnitude of vector

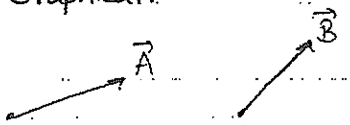
A → scalar

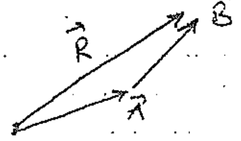
Magnitude } equal ⇒ Vectors are equal
Direction }

$\vec{A} = \vec{B} \Rightarrow |\vec{A}| = |\vec{B}|$

Adding Vectors: → Graphical
→ Algebraic

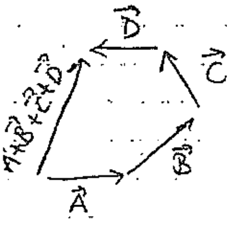
Graphical:



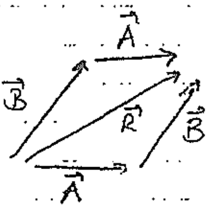


(tip to tail / head to tail)

$\vec{R} = \vec{A} + \vec{B}$



Row of addition $\vec{A} + \vec{B} = \vec{B} + \vec{A}$



$\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$

To add vectors, they must be of same unit, same type of quantity.

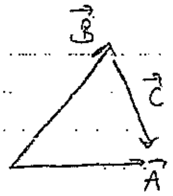
Negative of a vector:

$\vec{A}+(-\vec{A})=0$ Same magnitude, different direction.

Substracting Vectors:

1. $\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

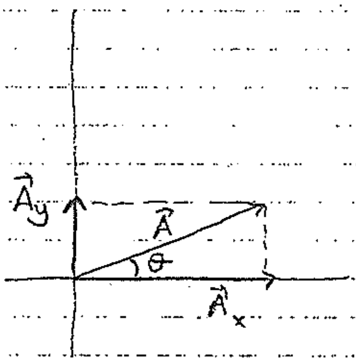
2.



$\vec{C}=\vec{A}-\vec{B}$

Algebraic:

Component of a vector:



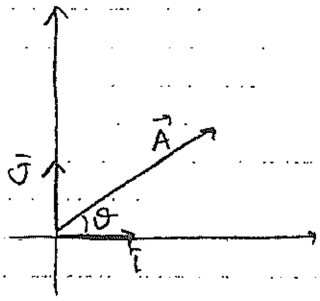
$\vec{A}=\vec{A}_x+\vec{A}_y$

$A=\sqrt{A_x^2+A_y^2}$

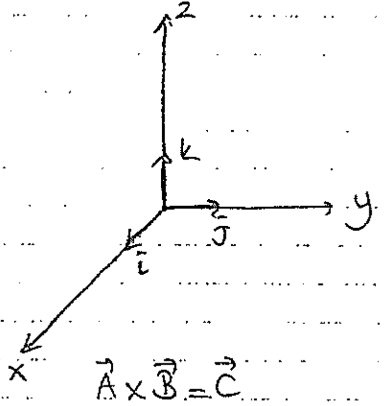
$\theta=\tan^{-1}\frac{A_y}{A_x}$

$A_x=A\cos\theta$ $A_y=A\sin\theta$

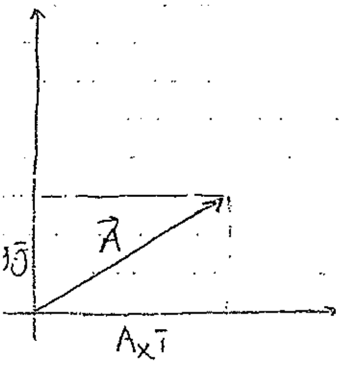
Unit vector: Dimensionless vector with a magnitude of exactly 1. Used to specify a direction and have no other physical significance. $\hat{i},\hat{j},\hat{k}$



$|\bar{i}|=|\bar{j}|=1$



$\vec{A}\times\vec{B}=\vec{C}$

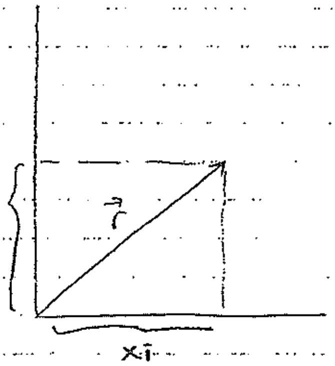


$$\vec{A}_x = \vec{A}_x \hat{i}$$

$$\vec{A}_y = \vec{A}_y \hat{j}$$

$$\vec{A} = \vec{A}_x \hat{i} + \vec{A}_y \hat{j}$$

Position vector:



$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{A} = \vec{A}_x \hat{i} + \vec{A}_y \hat{j}$$

$$\vec{B} = \vec{B}_x \hat{i} + \vec{B}_y \hat{j}$$

$$\vec{R} = \vec{A} + \vec{B} = (\vec{A}_x \hat{i} + \vec{A}_y \hat{j}) + (\vec{B}_x \hat{i} + \vec{B}_y \hat{j})$$

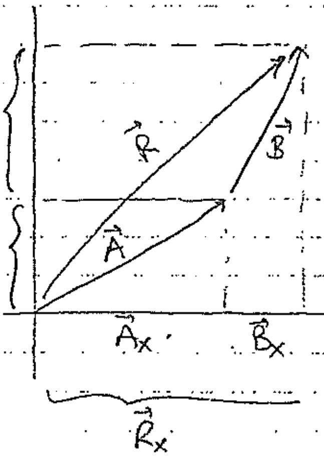
$$= (\vec{A}_x \hat{i} + \vec{B}_x \hat{i}) + (\vec{A}_y \hat{j} + \vec{B}_y \hat{j})$$

$$= (\vec{A}_x + \vec{B}_x)\hat{i} + (\vec{A}_y + \vec{B}_y)\hat{j}$$

$$\vec{R} = \vec{R}_x \hat{i} + \vec{R}_y \hat{j}$$

$$\vec{R}_x = \vec{A}_x + \vec{B}_x$$

$$\vec{R}_y = \vec{A}_y + \vec{B}_y$$



3/53 $\vec{R} = 2\hat{i} + \hat{j} + 3k$ find;

a. The magnitude of the x, y and z components.

$$R = \underset{\substack{\downarrow \\ 2.0}}{R_x \hat{i}} + \underset{\substack{\downarrow \\ 1.0}}{R_y \hat{j}} + \underset{\substack{\downarrow \\ 3.0}}{R_z k}$$

b. The magnitude of $\vec{R}$

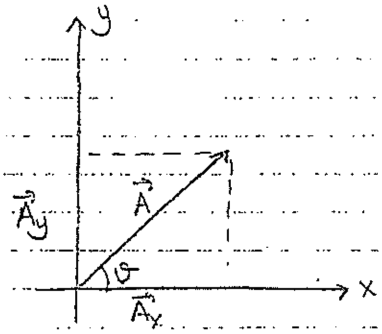
$$|\vec{R}| = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$|\vec{R}| = \sqrt{14} \text{ m}$$

c. The angles between $\vec{R}$ and the x, y and z axis.

$$\boxed{\cos\theta_x = \frac{R_x}{|\vec{R}|}}$$

$$\theta_x = \cos^{-1}\left(\frac{Rx}{|\vec{R}|}\right) \quad \theta_y = \cos^{-1}\left(\frac{Ry}{|\vec{R}|}\right) \quad \theta_z = \cos^{-1}\left(\frac{R_z}{|\vec{R}|}\right)$$



$$\vec{A} = \vec{A}_x + \vec{A}_y$$

$$A_x = |\vec{A}| \cos\vartheta$$

$$A_y = |\vec{A}| \sin\vartheta$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$

3/32

$\vec{A}$(−8,70 cm, 15 cm)

$\vec{B}$(13,20 cm, −6,60 cm)

If $\vec{A} + \vec{B} + 3\vec{C} = 0$ What are the components of $\vec{C}$?

$$\vec{C} = C_x \hat{i} + C_y \hat{j}$$

$$\vec{A} = -8.70\hat{i} + 15\hat{j}$$

$$\vec{B} = 13.20\hat{i} + (-6.60\hat{j})$$

$$3\vec{C} = -(\vec{A} + \vec{B})$$

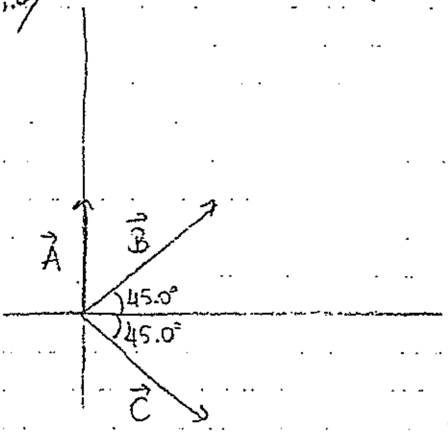
$$3(C_x \hat{i} + C_y \hat{j}) = -(4.50\hat{i} + 8.40\hat{j})$$

$C_x = -1.50$

$C_y = -2.80$

$\vec{C} = -1.50\hat{i} - 2.80\hat{j}$

.36/



$|\vec{A}|=2.0$ unit

$|\vec{B}|=40.0$ unit

$|\vec{C}|=30.0$ unit

Find.

a) the resultant in unit vector notation.

b) the magnitude and direction of the resultant displacement

$\vec{A}=|\vec{A}|\hat{j} \qquad \vec{A}=2.0\hat{j}$

$\vec{B}=|\vec{B}|.\cos\theta.\hat{i}+|\vec{B}|.\sin\theta.\hat{j} \qquad \vec{B}=40.0\cos45.0°\hat{i}+40.0.\sin45.0°\hat{j}$

$\vec{C}=|\vec{C}|.\cos\theta.\hat{i}-|\vec{C}|.\sin\theta.\hat{j} \qquad \vec{C}=30.0.\cos45.0°\hat{i}-30.0.\sin45.0°\hat{j}$

$\vec{R}=\vec{A}+\vec{B}+\vec{C}$

$\vec{R}=R_x\hat{i}+R_y\hat{j} \qquad R_x=B_x+C_x \qquad R_y=A_y+B_y+C_y$

$R_x=40.0.\cos45.0°+30.0.\cos45.0°=49,5$

$R_y=2.0+40.0.\sin45.0°-30.0.\sin45.0°=27,1$

$\vec{R}=R_x\hat{i}+R_y\hat{j}$

$\vec{R}=49,5\hat{i}+27,1\hat{j}$ //

$|R|=\sqrt{R_x^2+R_y^2}$

$|\vec{R}|=\sqrt{(49,5)^2+(27,1)^2}$

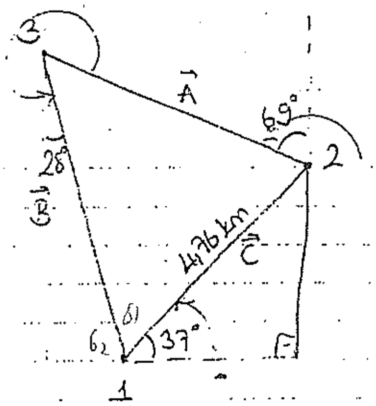
$|\vec{R}|=56,4$ unit

$\tan\theta=\dfrac{R_y}{R_x} \Rightarrow \theta=\tan^{-1}\left(\dfrac{R_y}{R_x}\right)$

$\theta=\tan^{-1}\left(\dfrac{27,1}{49,5}\right)$

$\theta=28,7°$ //

3.50 A ferry transports tourists ....



Calculate the distance between

a) The second and third islands.

b) The first and third islands

$\vec{C} = |\vec{C}| \cdot \cos 37{,}0° \hat{i} + |\vec{C}| \sin 37.0° \hat{j}$

for x components :

$4{,}76 \cdot \cos 37° + |\vec{A}| \cdot \cos 159\,0° + |\vec{B}| \cdot \cos 298° = 0$

$\boxed{3.80\ km - 0{,}934 A + 0{,}169 B = 0}$ ①

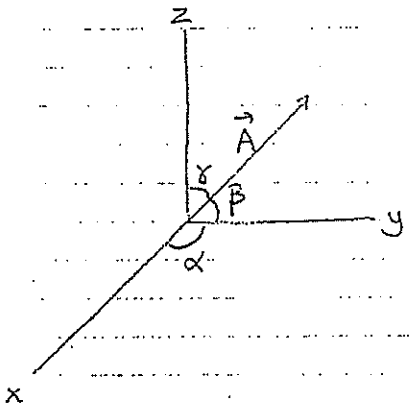
for y components.

$4{,}76 \cdot \sin 37 + |\hat{A}| \cdot \sin 159.0° + |\vec{B}| \cdot \sin 298° = 0$

$\boxed{2.86\ km + 0{,}358 A - 0{,}883 B = 0}$ ②

a) $A = 7{,}17\ km$

b) $B = 6{,}15\ km$



$\vec{A} = A_x \hat{i} \pm A_y \hat{j} + A_z \hat{k}$

$\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$

$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$

$\hat{i} . \hat{i} = 1$

$$\boxed{\vec{A}\cdot\vec{B} = |\vec{A}|\cdot|\vec{B}|\cdot\cos\alpha}$$

$$\vec{A}\cdot\hat{\imath} = A_x$$

$$\vec{A}\cdot\hat{\jmath} = A_y$$

$$\vec{A}\cdot\hat{k} = A_z$$

$$\hat{A} = \frac{A_x}{|\vec{A}|}\hat{\imath} + \frac{A_y}{|\vec{A}|}\hat{\jmath} + \frac{A_z}{|\vec{A}|}\hat{k}$$

$$\hat{A} = \frac{|\vec{A}|\cos\alpha}{\cancel{|\vec{A}|}}\hat{\imath} + \frac{|\vec{A}|\cos\beta}{\cancel{|\vec{A}|}}\hat{\jmath} + \frac{|\vec{A}|\cdot\cos\gamma}{\cancel{|\vec{A}|}}\hat{k}$$

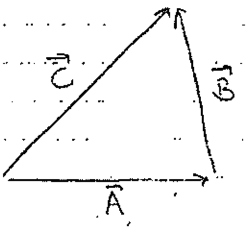
$$\hat{A} = \cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k}$$

$$|\vec{A}| = 1$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$(\alpha, \beta, \gamma) \leftarrow$ direction of cosines

★ Law of Cosines:



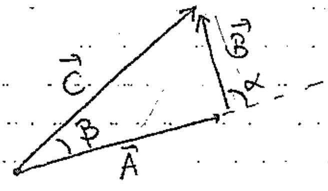
$$\vec{A}+\vec{B} = \vec{C}$$

$$\vec{B} = \vec{C}-\vec{A}$$

$$\vec{B}\cdot\vec{B} = (\vec{C}-\vec{A})(\vec{C}-\vec{A})$$

$$\boxed{B^2 = C^2 - 2AC + A^2} \leftarrow A^2 + C^2 - 2\cdot|\vec{A}|\cdot|\vec{C}|\cdot\cos\alpha$$

★ Law of Sines:



$$\vec{A}+\vec{B} = \vec{C}$$

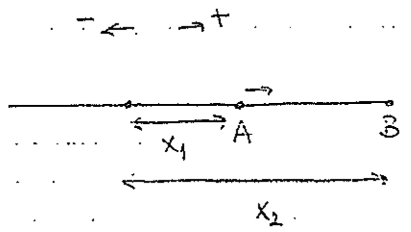
$$\vec{A}\times(\vec{A}+\vec{B}) = \vec{A}\times\vec{C} \qquad (\vec{A}\times\vec{A}) = 0$$

$$\underbrace{\cancel{(\vec{A}\times\vec{A})}}_{0} + (\vec{A}\times\vec{B}) = \vec{A}\times\vec{C} \qquad (\vec{A}\times\vec{B}) = AB\sin\alpha$$

$$\cancel{A}B\sin\alpha = \cancel{A}C\sin\beta$$

$$\boxed{B\sin\alpha = C\sin\beta}$$

#CHAPTER 2#



$\Delta x = x_2 - x_1$ $x_2 > x_1$ +

$x_2 < x_1$ −

Displacement is a vectorel quantity!

Kinematics:

Describes the motion while ignoring the external agents that might have caused or modified the motion. Along a straight line.

There are three types of motion:

- → Translational
- → Rotational
- → Vibrational

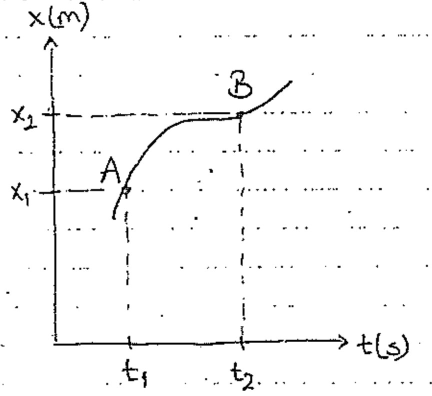
Particle Model:

A particle is a point like object has mass and infinitesimal size.

We will use the particle model.

Position:

The object's position is its location with respect to chosen reference point.



$$v_{x,avg} = \frac{\Delta x}{\Delta t}$$

$$v_{x,avg} = \frac{x_2 - x_1 \ (m)}{t_2 - t_1 \ (s)}$$ Unit of $v_{x,avg}$ is m/s

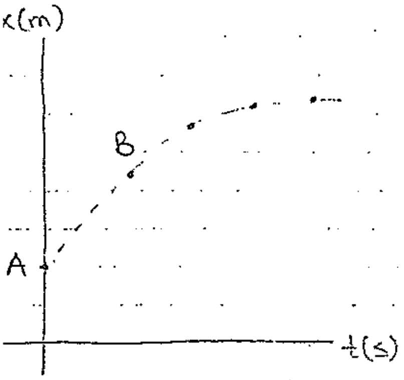
$$v_{x,avg} = \frac{x_f - x_i}{t_f - t_i}$$

Vector quantity

★ Average speed is a scalar quantity.

$v_{avg} = \frac{d}{t}$ → the total distance
↳ the total time

Instantaneous Velocity:



$v_{x,avg}$ → Average velocity
↓
the limit of average velocity

The limit of the average velocity as the time interval becomes infinitesimally short, or as time interval approaches to zero.

$$v_x = \lim_{\Delta t \to 0} \frac{x_f - x_i}{t_f - t_i}$$

$$\boxed{v_x = \frac{dx}{dt}}$$

The instantaneous velocity can be positive, negative or zero.

Instantaneous Speed:

"magnitude of instantaneous velocity"

"velocity" and "speed" will indicate instantaneous values.

Average will be used when the average velocity or average speed is indicated.

Model: A particle under constant velocity.

$v_x = v_{x,avg}$

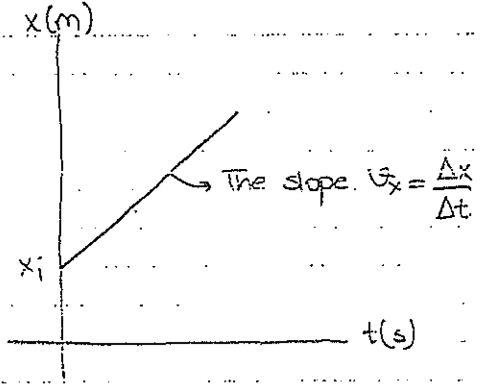
↗ is also the slope of the line in the position-time graph

$v_x = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$ OR $x_f - x_i = v_x . \Delta t$

$$\boxed{x_f = x_i + v_x . \Delta t}$$

$t_i = 0 \quad t_f = t$

$\boxed{x_f = x_i + v_x . t}$ for constant velocity (v_x)



Average Speed:

Speed is a scalar quantity.

Has the same units as velocity.

Defined as total distance / total time $\vartheta_{avg} = \frac{d}{t}$

★ The slope has units (unless both axes have the same units.)

Average Acceleration:

$$a_{x;avg} = \frac{\Delta \vartheta_x}{\Delta t} = \frac{\vartheta_{xf} - \vartheta_{xi}}{t_f - t_i}$$

Dimensions are L/T^2 $(L/T \Rightarrow \vartheta)$

SI units are m/s^2

In one dimension, positive and negative can be used to indicate direction.

Instantaneous Acceleration:

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta \vartheta_x}{\Delta t} = \frac{d\vartheta_x}{dt} = \frac{d^2 x}{dt^2}$$

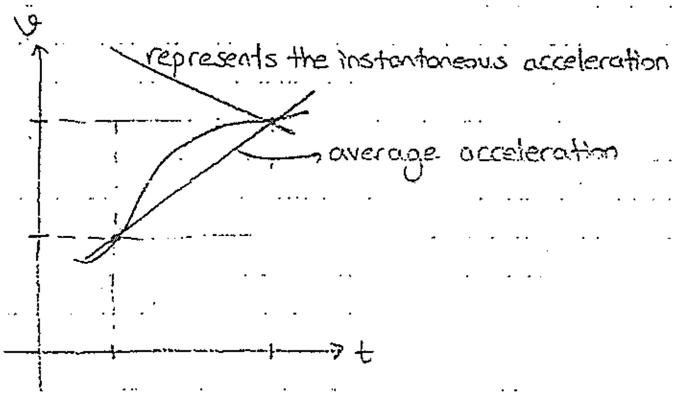
The term occeleration will mean instantaneous acceleration.

If average acceleration is wanted, the word average will be included.

$x(t) = 2t^3 + 3t^2 + 1$

$\vartheta_x = \frac{dx}{dt} = 6t^2 + 6t$

$a_x = \frac{d^2x}{dt^2} = 12t + 6$



When an object's velocity and acceleration are in the same direction, the object is eeding up.

When an object's velocity and acceleration are in the opposite direction, the object is wing down.

★ Negative acceleration does not necessarily mean that object is slowing down. (If the eleration and velocity are both negative, the object is speeding up.)

Kinematic Equations:

1. For constant a_x

$$v_{xf} = v_{xi} + a_x t$$

can determine an object's velocity at any time t when we know its initial velocity d its acceleration. (Does not give any information about displacement. Assumes $t_i = 0$ $t_f = t$)

$$a_x = a_{x,avg} \qquad a_{x,avg} = \frac{v_{xf} - v_{xi}}{\Delta t}$$

$$v_{xf} - v_{xi} = a_x t$$

$$\boxed{v_{xf} = v_{xi} + a_x t}$$

2. For constant acceleration

$$\boxed{v_{x,avg} = \frac{v_{xi} + v_{xf}}{2}}$$

This applies only in situations where the acceleration is constant!

3. For constant acceleration

$$\boxed{x_f = x_i + \vartheta_{x,avg}\cdot t = x_i + \frac{1}{2}(\vartheta_{xi} + \vartheta_{xf})\cdot t}$$

This gives you the position of the particle in terms of time and velocities.

Doesn't give you the acceleration.

4. For constant acceleration

$$x_f = x_i + \vartheta_{xi} t + \frac{1}{2} a_x t^2$$

Gives final position in terms of velocity and acceleration.

Does not tell you about final velocity.

$$x_f = x_i + \frac{1}{2}(\vartheta_{xi} + \underbrace{\vartheta_{xf}}_{\uparrow})t$$

$$\vartheta_{xf} = \vartheta_{xi} + a_x t$$

$$x_f = x_i + \frac{1}{2}(\vartheta_{xi} + \vartheta_{xi} + a_x t)t$$

$$\boxed{x_f = x_i + \vartheta_{xi} t + \frac{1}{2} a_x t^2}$$

5. For constant a,

$$\vartheta_{xf}^2 = \vartheta_{xi}^2 + 2a_x(x_f - x_i)$$

Gives final velocity in terms of acceleration and displacement.

Does not give any information about the time.

$$\boxed{\vartheta_{xf}^2 - \vartheta_{xi}^2 = 2a_x \Delta x}$$

$$x_f = x_i + \frac{1}{2}(\vartheta_{xi} + \vartheta_{xf})t \qquad x_f - x_i = \frac{1}{2}(\vartheta_{xi} + \vartheta_{xf})t$$

$$\vartheta_{xf} = \vartheta_{xi} + a_x t$$

$$t = \frac{\vartheta_{xf} - \vartheta_{xi}}{a_x}$$

$$x_f = x_i + \vartheta_{xi}\left(\frac{\vartheta_{xf} - \vartheta_{xi}}{a_x}\right) + \frac{1}{2} a_x \cdot \left(\frac{\vartheta_{xf} - \vartheta_{xi}}{a_x}\right)^2$$

$$x_f = x_i + \frac{\vartheta_{xi} \cdot \vartheta_{xf}}{a_x} - \frac{\vartheta_{xi}^2}{a_x} + \frac{1}{2} \cdot \frac{\vartheta_{xf}^2 - 2\vartheta_{xf}\vartheta_{xi} + \vartheta_{xi}^2}{a_x}$$

$$x_f = x_i + \cancel{\frac{\vartheta_{xi} \cdot \vartheta_{xf}}{a_x}} - \frac{\vartheta_{xi}^2}{a_x} + \frac{1}{2} \cdot \frac{\vartheta_{xf}^2}{a_x} - \frac{\vartheta_{xf} \cdot \vartheta_{xi}}{a_x} + \frac{1}{2} \cdot \frac{\vartheta_{xi}^2}{a_x}$$

$$x_f = x_i + \frac{1}{2}\left(\frac{\vartheta_{xf}^2 - \vartheta_{xi}^2}{a_x}\right)$$

$(x_f - x_i) \cdot 2a_x = v_{xf}^2 - v_{xi}^2$

$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i)$

$$\boxed{v_{xf}^2 = v_{xi}^2 + 2a_x \Delta x}$$

When $a=0$

$v_{xf} = v_{xi} = v_x$

$x_f = x_i + v_x t$

The constant acceleration model reduces to the constant velocity model.

→ Freely Falling Objects

 $|a_y| = -g$ $g = 9.80\ m/s^2$

altitude ↑ g ↑

g varies with latitude

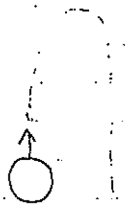
we'll neglect air resistance

With upward being positive initial velocity will be negative.

 $v_0 \neq 0$ $a=-g$

When you throw the object upward;

$\frac{1}{5} - \frac{3}{10}$



Initial velocity is upward, so positive.

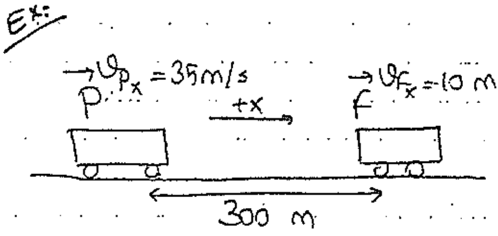
$\frac{2}{5} \cdot \frac{3}{10}$

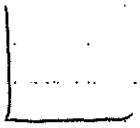
$v_0 \neq 0$ $a = -g$

$t^2 + 250t - 3000$

$a_y = -g = -9.80\ m/s^2$ everywhere in the motion $+\frac{1}{b}$ $wo\ t$

Ex:





$35t + \frac{3}{5} \cdot \frac{1}{2} t^2 = 10t + \frac{1}{2} \cdot \frac{2}{5} t^2$

$25t + \frac{t^2}{10} = 300$ $t^2 - 250t + 3000$

When the distance between them is 300 m, the engineers become aware of the danger. avoid a collusion train P slows down with an acceleration of $a_P = 0.60\ m/s^2$ and train speeds up with an acceleration $a_F = 0.40\ m/s^2$. Find the time at which the trains ride from the moment they notice each other.

$$\left.\begin{array}{l}\Delta x_p = v_{p_0} t + \frac{1}{2} a_p t^2 \\ \Delta x_f = v_{f_0} t + \frac{1}{2} a_f t^2\end{array}\right\} \Delta x_p = 300 + \Delta x_f$$

$\Delta x_p = 300 + \Delta x_f$

$35.t - \frac{1}{2}.(0,6).t^2 = 300 + 10.t + \frac{1}{2}.(0,4).t^2$

$35t - 0,3t^2 = 300 + 10t + 0,2t^2$

$0,5t^2 - 25t + 300 = 0$

$t^2 - 50t + 600 = 0$

$t \quad -30 \qquad t = 30s \quad \boxed{t = 20s}$

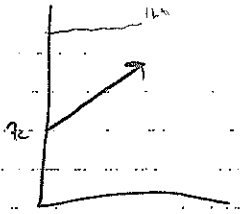
$t \quad -20$

Ex: A car traveling on a long straight highway at a constant speed of 144 km/h passes a police motorcycle which moves at 72 km/h but immediately accelerates at a constant rate of 20 m/s² to catch the speeding car.

a) How long will it take the police motorcycle to catch the speeding car?

$\rightarrow v_p$

$\rightarrow v_c$



$v_{p_x} = 72\ km/h \longrightarrow \overset{2}{\cancel{72}}\,\frac{\cancel{1000}}{\cancel{3600}} = 20\ m/s$

$v_{c_x} = 144\ km/h \longrightarrow \overset{4}{\cancel{144}}\,\frac{\cancel{1000}}{\cancel{3600}} = 40\ m/s$

$144t = 72t + 10t^2$

The car moves at constant speed $\Rightarrow x_c = v_{c_0} t + 0$

$x_p = v_{p_0} t + \frac{1}{2} a t^2$

$\boxed{x_c = x_p} \rightarrow$ catches

$40.t + 0 = 20.t + \frac{1}{2}.2.t^2$

$20t = t^2$

$\underline{\underline{t = 20s}}$

b) How fast will the police motorcycle be traveling when it reaches the speeding car? Give it in km/h

$\vartheta_{P_0} = 72\ km/h = 20\ m/s$

$\vartheta_{P_f} = \vartheta_{P_0} + at$

$= 20 + 2.20$

$= 60\ m/s \Rightarrow 6\cancel{0}\cdot\frac{36\cancel{00}}{1\cancel{000}} = 216\ km/h$

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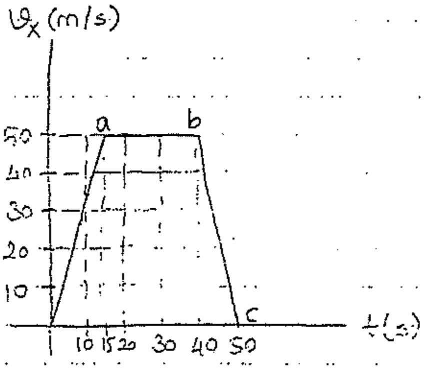


a) Calculate total distance

$A = \frac{1}{2}\cdot 15\cdot 50 + 25\cdot 50 + \frac{1}{2}\cdot 10\cdot 50 = 25\cdot 15 + 25\cdot 50 + 25\cdot 10 = 25\cdot 75 = \underline{\underline{1875\ m}}$

b) Acceleration

$a_{o-a} = \frac{50}{15} = \frac{10}{3}\ m/s^2$

$a_{a-b} = 0\ m/s^2$

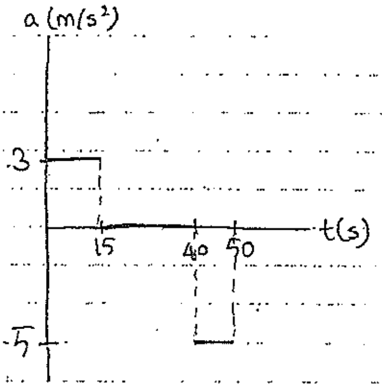
$a_{b-c} = -\frac{50}{10} = -5\ m/s^2$

c) Draw a graph of its acceleration versus time between t=0 and t=50 s

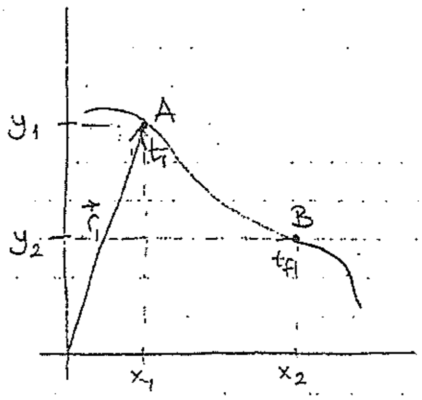
Oa $0 \leq t \leq 15\,s$ $a_1 = \frac{\Delta\vartheta}{\Delta t} = \frac{50-0}{15-0} = \frac{10}{3} = 3.3\ m/s^2$

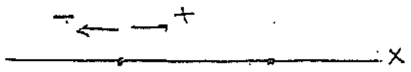
ab $15 \leq t \leq 40\,s$ $a_2 = 0$

bc $40 \leq t \leq 50\,s$ $a_3 = \frac{\Delta\vartheta}{\Delta t} = \frac{0-50}{50-40} = -5\ m/s^2$



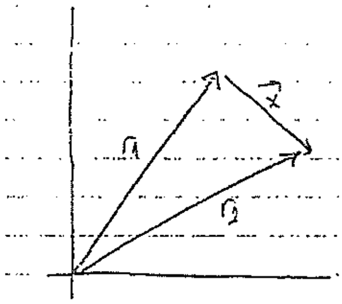
#CHAPTER 4#





$\Delta x = x_f - x_i$
↓
displacement

$$\boxed{\vec{r} = x\hat{i} + y\hat{j}}$$



$$\boxed{\Delta\vec{r} = \vec{r_f} - \vec{r_i}}$$

$\vec{v}_{avg} = \frac{\Delta\vec{x}}{\Delta t}$ displacement

$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$ displacement vector

Instantaneous Velocity:

$$v = \lim_{\Delta t \to 0} \frac{\Delta\vec{x}}{\Delta t} = \frac{d\vec{x}}{dt}$$

As the time interval becomes smaller, the direction of the displacement approaches that of the line tangent to the curve.

Average Acceleration:

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

Instantaneous Acceleration:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Kinematic Equations for Two Dimensional Motion:

$\rightarrow \vec{r} = x\hat{i} + y\hat{j}$

$$\vec{v} = \frac{d\vec{x}}{dt} = v_x\hat{i} + v_y\hat{j}$$

Since acceleration is constant, we can also find an expression for the velocity as a action of time.

$$\vec{v}_f = \vec{v}_i + \vec{a}t$$

$\rightarrow \vec{r}_f = \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{a}t^2$

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\frac{dx}{dt}}_{v_x}\hat{i} + \underbrace{\frac{dy}{dt}}_{v_y}\hat{j}$$

Ex: An object moving in a plane and the coordinate of its position is given as $x = t^2 + 2t$ x(m) t(s)
$y = 3t$

a) Find its position at time t=1 s

$$\vec{r}_1 = x\hat{i} + y\hat{j}$$

$$\vec{r}_1 = (t^2+2t)\hat{i} + (3t)\hat{j}$$

$$\vec{r}_1(t=1s) = 3\hat{i} + 3\hat{j}$$

b) Find its initial velocity at t=1 s

$$\vec{r} = (t^2+2t)\hat{i} + (3t)\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (2t+2)\hat{i} + 3\hat{j}$$

$$\vec{v}(t=1s) = 4\hat{i} + 3\hat{j}$$

c) Find its acceleration

$$\vec{v} = (2t+2)\hat{i} + 3\hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 2\hat{i}$$

$\vec{a} = 2\hat{i}$ $\vec{a} = \underset{2\,m/s^2}{\boxed{a_x}}\hat{i} + \underset{0}{\boxed{a_y}}\hat{j}$

d) Find its position and velocity at t=2 s

$\vec{r} = (t^2+2t)\hat{\imath} + (3t)\hat{\jmath}$

$\vec{r}\,(t=2s) = 8\hat{\imath} + 6\hat{\jmath}$ //

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$\vec{\vartheta} = (2t+2)\hat{\imath} + 3\hat{\jmath}$

$\vec{\vartheta}(t=2s) = 6\hat{\imath} + 3\hat{\jmath}$ //

★ $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ $a_x = \text{constant}$
$a_y = \text{constant}$

$$\vec{\vartheta} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$$

$\vec{\vartheta} = \vartheta_x\hat{\imath} + \vartheta_y\hat{\jmath}$ $\vartheta_x = \vartheta_{x0} + a_x t$ $\vartheta_y = \vartheta_{y0} + a_y t$

$$\vec{\vartheta} = (\vartheta_{x0} + a_x t)\hat{\imath} + (\vartheta_{y0} + a_y t)\hat{\jmath} = \vartheta_{x0}\hat{\imath} + a_x t\hat{\imath} + \vartheta_{y0}\hat{\jmath} + a_y t\hat{\jmath}$$
$$= (\vartheta_{x0}\hat{\imath} + \vartheta_{y0}\hat{\jmath}) + (a_x\hat{\imath} + a_y\hat{\jmath})t$$

$x = x_0 + \vartheta_{0x}t + \frac{1}{2}a_x t^2$

$y = y_0 + \vartheta_{0y}t + \frac{1}{2}a_y t^2$

$\vec{r} = x\hat{\imath} + y\hat{\jmath}$

$$\vec{r} = (x_0 + \vartheta_{0x}t + \tfrac{1}{2}a_x t^2)\hat{\imath} + (y_0 + \vartheta_{0y}t + \tfrac{1}{2}a_y t^2)\hat{\jmath}$$

$$\vec{r} = (x_0\hat{\imath} + y_0\hat{\jmath}) + (\vartheta_{0x}\hat{\imath} + \vartheta_{0y}\hat{\jmath}) + \frac{1}{2}(a_x\hat{\imath} + a_y\hat{\jmath})t^2$$

Ex: A particle leaves the origin with an initial velocity $\vec{\vartheta}_0 = 3.0\hat{\imath}$ (m/s) and experiences a constant acceleration. $\vec{a} = -1.0\hat{\imath} - 0{,}5\hat{\jmath}$ (m/s²)

What is the velocity of the particle when it reaches maximum x coordinate.

$\vartheta = \vartheta_0 + at$

$3\hat{\imath} - \hat{\imath}t$

$\vartheta_x = \vartheta_{0x}\hat{\imath} + a_x t$ $t=?$

$\vartheta_x = 3\hat{\imath} - 1.0t$

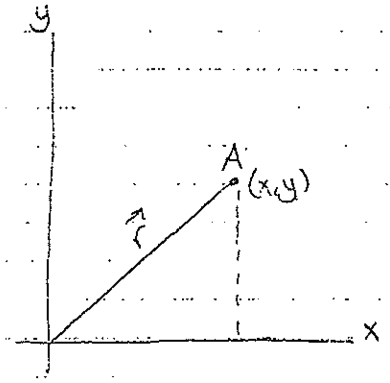
$\vartheta_y = 0 - 0.5t$

When the particle reaches max $t = 3\,s$ $\vartheta_y = -1{,}5$ m/s

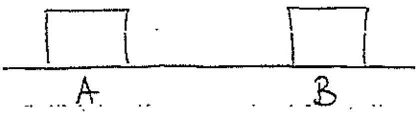
Where is the particle at that time?

The position vector

$\vec{r} = x\hat{i} + y\hat{j}$



$v_f = v_i + a_x t$



$v_{xf} = v_{xi} + a_x t$

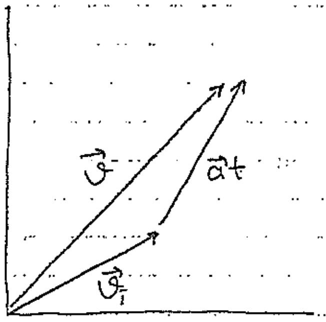
$v_{yf} = v_{yi} + a_y t$

$\vec{v} = v_x\hat{i} + v_y\hat{j}$

$\vec{v} = (v_{xi} + a_x t)\hat{i} + (v_{yi} + a_y t)\hat{j}$

$$\vec{v} = \underbrace{(v_{xi}\hat{i} + v_{yi}\hat{j})}_{\vec{v}_i} + \underbrace{(a_x\hat{i} + a_y\hat{j})}_{\vec{a}}t$$

$\vec{v} = \vec{v}_i + \vec{a}t$



$\vec{v} = \vec{v}_i + \vec{a}t$

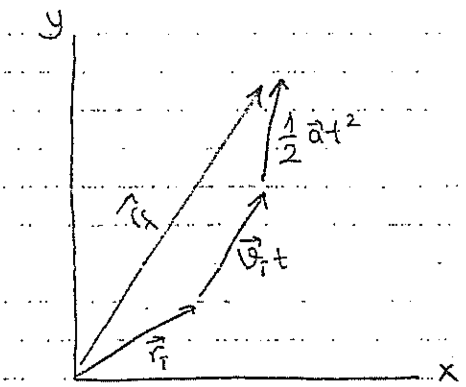
$\vec{r} = x\hat{i} + y\hat{j}$

$x_f = x_i + v_{x_i} t + \frac{1}{2} a_x t^2$

$y_f = y_i + v_{y_i} + \frac{1}{2} a_y t^2$

$\vec{r} = (x_i + v_{x_i} t + \frac{1}{2} a_x t^2)\hat{i} + (y_i + v_{y_i} t + \frac{1}{2} a_y t^2)\hat{j}$

$$\vec{r} = \underbrace{(x_i \hat{i} + y_i \hat{j})}_{\vec{r}_i} + \underbrace{(v_{x_i} \hat{i} + v_{y_i} \hat{j})}_{\vec{v}_i} + \frac{1}{2} \underbrace{(a_x \hat{i} + a_y \hat{j})}_{\vec{a}} t^2$$



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A package is dropped from a helicopter descending steadily at a speed v_i (at time $t=0$)

a) What is the speed of the package in terms of v_i, g and t?

$\uparrow +$

$\downarrow -$

$a_y = -g$

$v_p = -v_i - gt$

$v_p = -(v_i + gt)$

$t > 0$

$|v_p| = v_i + gt$

b) what vertical distance d is it from the helicopter in terms of g and t?

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y_H, y_P

$$y = y_i + v_{yi} t - \frac{1}{2} g t^2$$

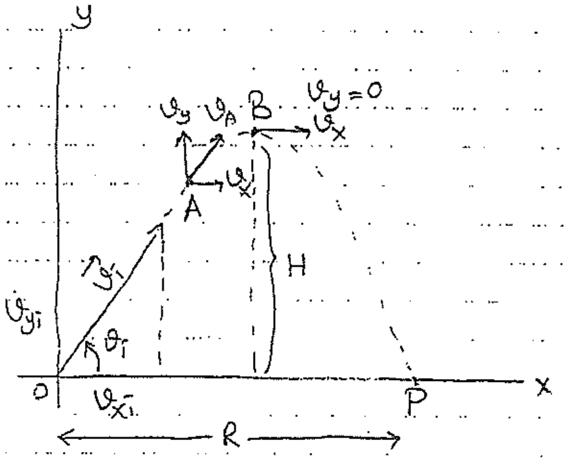
$$\downarrow$$

$$y_P = H - v_i t - \frac{1}{2} g t^2$$

$$y_H = H - v_i t$$

$$d = y_H - y_P = \frac{1}{2} g t^2$$

Projectile Motion:



$v_{xi} = v_i \cos\theta_i$ ... $a_x = 0$

$v_{yi} = v_i \sin\theta_i$ .... $a_y = -g$

$$\vec{v}_i = v_{xi}\,\hat{i} + v_{yi}\,\hat{j}$$

★ v_x doesn't change.

$$v_x = v_{xi}$$

$$v_x = v_i \cos\theta_i$$

$$v_y = v_{yi} - gt$$

$$v_y = v_i \sin\theta_i - gt$$

$$t_B = \frac{v_{yi}}{g} = \frac{v_i \sin\theta_i}{g}$$

$$t_P = 2t_B = 2\frac{v_i \sin\theta_i}{g}$$

$$x = v_x t_P = v_i \cos\theta_i \left(2 \cdot \frac{v_i \cdot \sin\theta_i}{g}\right)$$

$$x = R = \frac{v_i^2 \cdot 2 \cdot (\sin\theta_i \cdot \cos\theta_i)}{g}$$

$$\boxed{R = \frac{v_i^2 \cdot \sin 2\theta}{g}} \quad (\sin 2\theta = 2\sin\theta\cos\theta)$$

$$\begin{cases} x = x_i + v_{xi} t \\ y = y_i + v_{yi} t - \frac{1}{2} g t^2 \end{cases}$$

$$x - x_i = v_i . \cos\theta_i . t$$

$$y - y_i = v_{yi} . \sin\theta_i . t - \frac{1}{2} g t^2$$

$$\boxed{t = \frac{x - x_i}{v_i . \cos\theta_i}}$$

$$y = ax^2 + bx$$

$$\boxed{y = (\tan\theta) x - \left(\frac{1}{2} . \frac{g}{v_i^2 . \cos^2\theta_i} \right) . x^2}$$

The trajectory of a projectile is a parabola.

$$v_A = \sqrt{v_x^2 + v_y^2}$$

$$y_i = 0$$

$$y = v_{yi} t - \frac{1}{2} g t^2$$

$$v_{yi} = v_i . \sin\theta_i$$

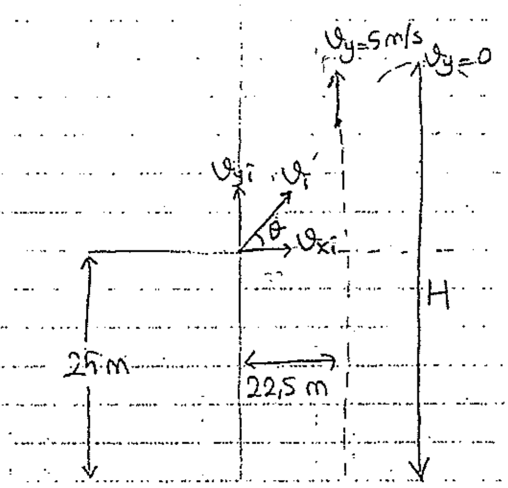
$$v_y = v_i - gt$$

$$0 = v_i \sin\theta_i - gt \Rightarrow t_B = \frac{v_i . \sin\theta_i}{g}$$

$$H = v_i . \sin\theta_i \left(\frac{v_i . \sin\theta_i}{g} \right) - \frac{1}{2} . g . \left(\frac{v_i . \sin\theta_i}{g} \right)^2$$

$$H = \frac{v_i^2 . \sin^2\theta_i}{2g}$$

Ex:



At time t=0 an object is thrown from the roof of a building 25 m above the ground. 1,5 s after it is thrown, the vertical component of the $\vec{v}$ is $v_y = 5$ m/s. The horizontal component of the displacement is 22,5 m. $x =$

$V_{xi} = \frac{3}{2} = \frac{45}{2}$ $V_{xi} = 15$ m/s

5 s $V_{oy} = \frac{3}{2} . 10$ [illegible] [illegible]

a) Calculate the magnitude of the initial velocity of the object. what is the tangent f the angle the initial velocity makes with the horizontal?

$v_y = v_{yi} - gt$

$v_{yi} = v_y + gt$

$v_{yi} = 5 + 10.1{,}5 = 20 \text{ m/s}$

$x_f = x_i + v_{xi} . t$

$v_{xi} = \frac{x_f - x_i}{t} = \frac{22{,}5 - 0}{1{,}5} = 15 \text{ m/s}$

$v_i = \sqrt{v_{xi}^2 + v_{yi}^2} = 25 \text{ m/s}$

$\tan\theta = \frac{v_{yi}}{v_{xi}} = \frac{20}{15} = \frac{4}{3} \quad \left(\theta_i = \tan^{-1}\left(\frac{v_{yi}}{v_{xi}}\right)\right)$

b) Calculate the maximum height above the ground the object reaches.

$H = y_{max} \quad y_{max} = y + 25 \text{ m}$

$v_y = v_{yi} - gt$

$\underset{0}{\underset{\downarrow}{v_y^2}} = v_{yi}^2 - 2g\Delta y$

$\Delta y = \frac{v_{yi}^2}{2g} = \frac{20^2}{2.10} = 20 \text{ m}$

$H = 20 + 25 = \boxed{45 \text{ m}}$

c) Calculate the total time of flight for the object to hit the ground.

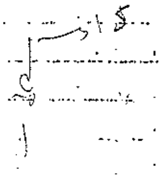
$y = y_0 + v_{yi}t - \frac{1}{2}gt^2$

$-25 = 20t - \frac{1}{2} . 10 . t^2$

$5t^2 - 20t - 25 = 0$

$\boxed{t = 5 \text{ s}}$

d) Calculate the velocity of the object just before it hits the ground. Express the answer the unit vector notation.



$v_y = v_{yi} - gt = 20 - 10.5 = -30 \text{ m/s}$

$v_x = v_{xi} = 15 \text{ m/s}$

$\vec{v}=v_x\hat{i}+v_y\hat{j}$

$\vec{v}=15\hat{i}-30\hat{j}$ (m/s)

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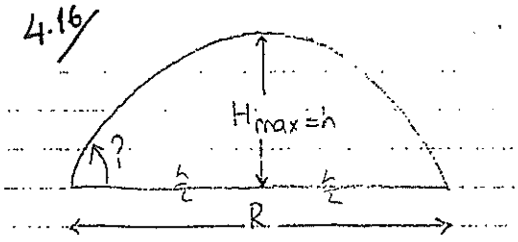


Level ground.

R=h

a) At what angle θ is the rock thrown?

$$R=h=\underbrace{\frac{\cancel{v_i^2}\cdot\sin^2\theta}{2\cancel{g}}}_{h}=\underbrace{\frac{\cancel{v_i^2}\cdot\sin2\theta}{\cancel{g}}}_{R}$$

$\sin^2\theta=2\sin2\theta$

$\sin^{\cancel{2}}\theta=2\cdot2\cdot\cancel{\sin\theta}\cdot\cos\theta$

$\tan\theta=4$

$\theta_i=\tan^{-1}(4)=76.0°$

b) In terms of its original range R, what is the range R_{max}?

$$R_{max}=\frac{v_i^2\cdot\sin2\theta_i}{g}\quad \theta_i=45° \Rightarrow R_{max}=\frac{v_i^2}{g}$$

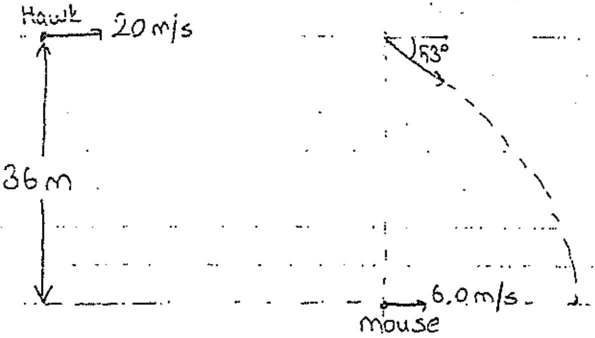
R_{max} in terms of R:

$$R=\frac{v_i^2\cdot\sin(2.76{,}0°)}{g}$$

$R=R_{max}\cdot\sin(2.76{,}0°)$

$$R_{max}=\frac{R}{\sin(2.76{,}0°)}$$

Ex) A hawk is flying.



Assuming that the hawk moves as a free body after it dives,

a) Find the initial diving velocity of the hawk. Give it in unit vector notation.

b) Where does the hawk catch the mouse relative to the point where it sees the use first time. Give it in unit vector notation.

c) What is the velocity of the hawk when it captures the mouse? Give it in unit ctor notation.

$h = 5 t^2$

$t^2 = \frac{36}{5}$

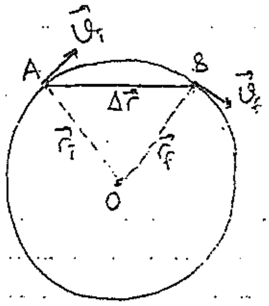
$t = \frac{6\sqrt{5}}{5}$

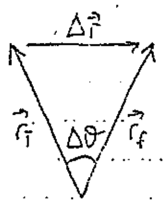
$V = at$

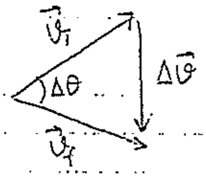
$\frac{6\sqrt{5}}{5} \cdot 10$ $\boxed{12\sqrt{5}}$

$1.2\sqrt{5}$

#UNIFORM CIRCULAR MOTION

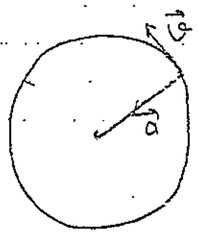






Centripetal Acceleration:

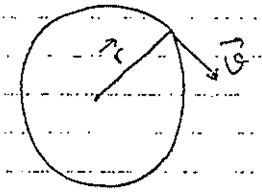
The acceleration is always perpendicular to the velocity.



$\underline{\underline{a_c}}$

$\frac{\Delta v}{v} = \frac{\Delta r}{r}$ $\boxed{\Delta\vec{v} = \left(\frac{\Delta\vec{r}}{r}\right) v}$

And the top view



$r = r_i = r_f$

$v = v_i = v_f$

$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{v \cdot \Delta\vec{r}}{r \cdot \Delta t}$

$\lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{dr}{dt}$

a_c ⇒ centripetal acceleration

$\boxed{a_c = \frac{v^2}{r}}$

T → the period

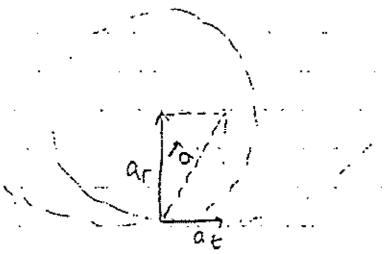
2πr → the circumference

$\boxed{v = \frac{2\pi r}{T}}$ $\boxed{T = \frac{2\pi r}{v}}$

f → the frequency $\boxed{f = \frac{1}{T}}$

Tangential Acceleration:

The magnitude of the velocity could also be changing. In this case, there would be a ngential acceleration.



→ Tangential acceleration causes the change in the speed of the particle.

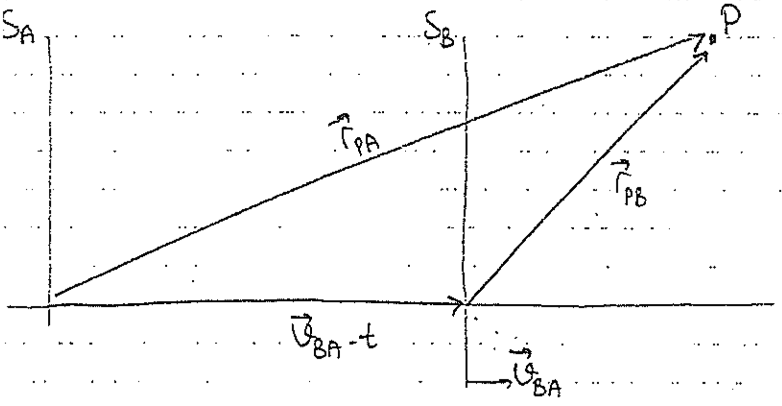
→ The radial acceleration comes from the change in the velocity vector.

$$a_t = \left| \frac{dv}{dt} \right|$$

→ The radial acceleration $a_r = -a_c = \frac{v^2}{r}$

→ The magnitude of acceleration $a = \sqrt{a_t^2 + a_r^2}$

Relative Velocity:



$$\boxed{\vec{r}_{PA} = \vec{r}_{PB} + \vec{v}_{BA}\cdot t}$$

derivative

$$\frac{d\vec{r}_{PA}}{dt} = \frac{d\vec{r}_{PB}}{dt} + \vec{v}_{BA}$$

$$\vec{u}_{PA} = \vec{u}_{PB} + \vec{v}_{BA}$$

Taking derivative

$$\frac{d\vec{u}_{PA}}{dt} = \frac{d\vec{u}_{PB}}{dt} + \overset{\text{constant}}{\left(\frac{d\vec{v}_{BA}}{dt}\right)} = 0 \Rightarrow \boxed{\vec{a}_{PA} = \vec{a}_{PB}}$$

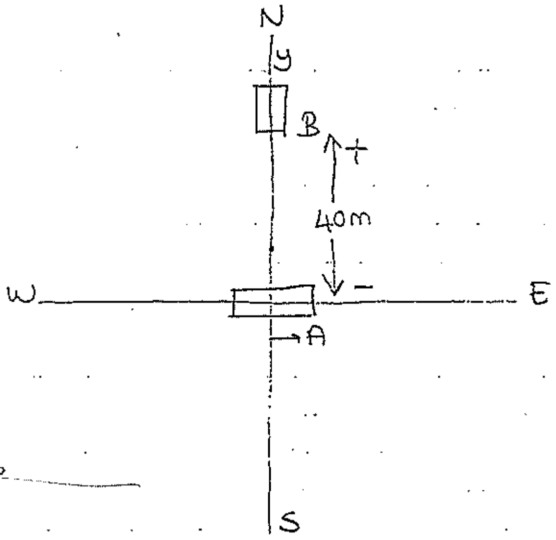
The first subscript represents what is being observed.

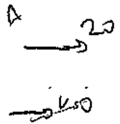
The second subscript represents who is doing the observing.

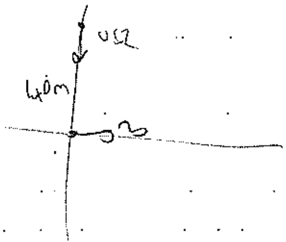
$r_{PA} \Rightarrow$ (P relative to A)

EX:

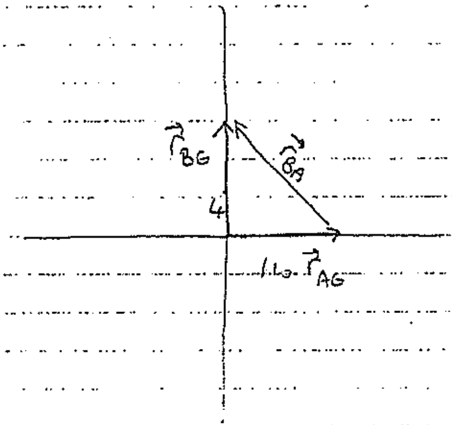
Car A is travelling due east at 20 m/s, car B starts from rest, 40 m north of the intersection and moves with an acceleration of 2,0 m/s².







a) What is the position of car B relative to car A, 6.0 s after car A crosses the intersection? Write in unit vector notation.



At t=6 s

$\vec{r}_{AG} = 20.(6,0) = 120\hat{\imath}\ (m)$

$\vec{r}_{BG} = 40 + \frac{1}{2}.(-2).6^2 = 4\hat{\jmath}\ (m)$

$\vec{r}_{BA} = \vec{r}_{BG} - \vec{r}_{AG} = 4\hat{\jmath} - 120\hat{\imath}\ (m)$

$\frac{1}{2}at^2$

b) What is the relative velocity of car B, relative to car A at t=6.0 s after car A crosses the intersections? Write it in unit vector notation.

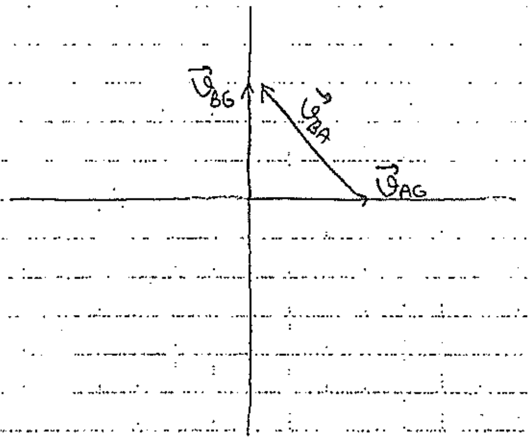
At t=6.0 s

$\vec{\vartheta}_{AG} = 20\hat{\imath}\ (m/s)$

$\vec{\vartheta}_{BG} = 2\hat{\jmath}.6 = 12\hat{\jmath}\ (m/s)$

$\vec{\vartheta}_{BA} = \vec{\vartheta}_{BG} - \vec{\vartheta}_{AG}$

$\vec{\vartheta}_{BA} = 12\hat{\jmath} - 20\hat{\imath}\ (m/s)$ //



c) What is the acceleration of the car B relative to car A at time t=6.0s after car crosses the intersections? Write it in unit vector notation.

$\vec{a}_{BA} = -2\hat{j}$ m/s²

#THE LAWS OF MOTION#

Why the motion of an object changes? Two main factors need to be addressed to answer.

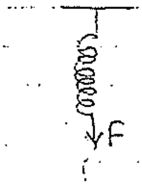
→ Forces acting on the object

→ The mass of the object

A force is that which causes an acceleration.

Fundamental forces:

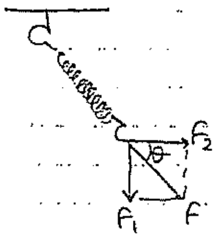
- Gravitational force
- Electromagnetic forces
- Nuclear force
- Weak forces



A spring can be used to calibrate the magnitude of a force.

Doubling the force causes double the reading on the spring.

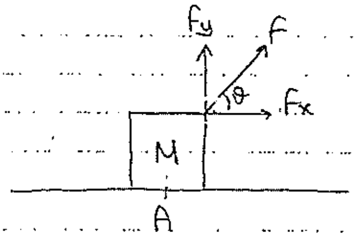
Vector Nature of Forces:



The forces are applied perpendicularly to each other.

The resultant (or net) force is the hypotenuse.

Forces are vectors, so you must use the rules for vector addition to find the net force acting on an object.



$\vec{F}$ → vectorel quantity

magnitude, direction

$\vec{F} = F_x\hat{i} + F_y\hat{j}$

$F_x = F.\cos\theta$ $F_y = F.\sin\theta$

Newton's first Law:

If an object does not interact with other objects, it is possible to identify a reference frame in which the object has zero acceleration.

Inertial Frames:

Any reference frame that moves with constant velocity relative to an inertial frame is itself an inertial frame.
↳ doğrusal hareket

If you accelerate relative to an object in an inertial frame, you are observing the object from a non-inertial reference frame.

In the absence of inertial forces, when viewed from an inertial reference frame, an object at rest remains at rest and object in motion continues in motion with a constant velocity.

Newton's first Law explains what happens in the absence of a force.

Inertia and Mass:

The tendency of an object to resist any attempt to change its velocity is called inertia.
↳ eğilim

Mass is the property of an object that specifies how much resistance an object exhibits to changes in its velocity.

$F_1 \longrightarrow \boxed{m_1}$ $\quad$ $F_2 \longrightarrow \boxed{m_2}$

$F \propto a$

If $F_1 = F_2$ $\quad$ $m_1 a_1 = m_2 a_2$

$\boxed{\frac{m_1}{m_2} = \frac{a_2}{a_1}} \rightarrow$ inversely proportional

- ✱ Mass is an inherent property of an object.
- ✱ Mass is independent of the object's surroundings.
- ✱ Mass is independent of the method used to measure it.
- ✱ Mass is a scalar quantity. Obeys the rules of ordinary arithmetics.

Mass vs. Weight:

Weight is equal to the magnitude of the gravitational force exerted on object.

Newton's Second Law:

$\vec{a} \propto \frac{\Sigma \vec{F}}{m} \rightarrow \Sigma \vec{F} = m.\vec{a}$

$\Sigma \vec{F}$ is the net force acting on the object.

$\Sigma F_x = m a_x$
and (in 3 dimension) $\Sigma F_z = m a_z$
$\Sigma F_y = m a_y$

The unit of $\vec{F}$: The SI unit of $\vec{F}$ is $1 \ kg \cdot m/s^2 = 1$ Newton (1N)

$1 \ lb = 1 \ slug.ft/s^2$

$1 N \sim \frac{1}{4} lb$

The gravitational force:

$\vec{F}_g = m\vec{g}$

The magnitude of the gravitational force:

$w \rightarrow$ weight of the object $\Rightarrow$ It varies with location.

$F_g = w = mg$

☆ The weight is not an inherent property of the object.

$\nearrow$ Gravitational mass $(\vec{F}_g = m\vec{g})$
eylemsizlik
$\searrow$ Inertial mass $(\vec{F} = m\vec{a})$
$\}$ Magnitudes are the same.

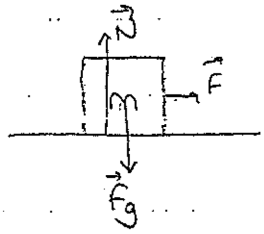
In Newton's Law, the mass is the inertial mass, and measures the resistance to a ange in the object's motion.

On the other hand, the gravitational mass measures the attraction between an object d Earth.

Experiments show that the magnitude of inertial mass is equal to the magnitude gravitational mass.

second

Newton's Third Law:



(There is no friction between the object and the surface.)

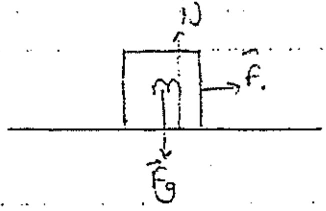
$\vec{N}, \vec{F_g}, \vec{F}$ are the forces acting on the object when it's in motion.

In Equilibrium (a=0)

$\Sigma\vec{F} = m\vec{a} \quad \vec{a} = 0$

$\Sigma F_x = ma_x \quad a_x = 0$

$\Sigma F_y = ma_y \quad a_y = 0$



x-direction:

$\Sigma\vec{F} = m\vec{a}$

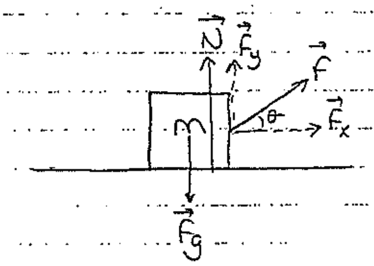
y-direction:

$\Sigma F_y = N - F_g = 0$

$a_y = 0$

$N = F_g \quad N = mg$

Weight $= mg \quad \boxed{W = mg}$



$F_x = F\cos\theta$

$F_y = F\sin\theta$

$\Sigma\vec{F} = m\vec{a}$

$\Sigma F_x = ma_x \longrightarrow \Sigma F_x = ma \quad a_x = 0$

$\Sigma F_y = ma_y \longrightarrow \Sigma F_y = 0 \quad a_y = 0$

x-direction:

$F\cos\theta = ma$

y-direction:

$N = mg - F\sin\theta$

$N + F_y - mg = 0$

The model is the particle in equilibrium model.

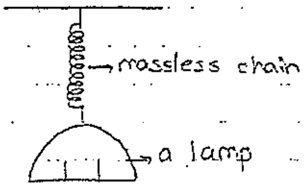
Mathematically, the net force acting on the object is zero.

$\Sigma \vec{F} = 0$

$\Sigma F_x = 0$

$\Sigma F_y = 0$

Ex:



In equilibrium;

$\vec{T} = \vec{F}_g$

$\downarrow$

$\boxed{T = mg}$

In equilibrium;

$\Sigma F_y = 0$

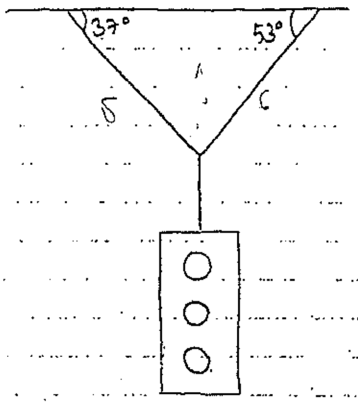
$\uparrow \; \Sigma F_y = T - F_g = 0$

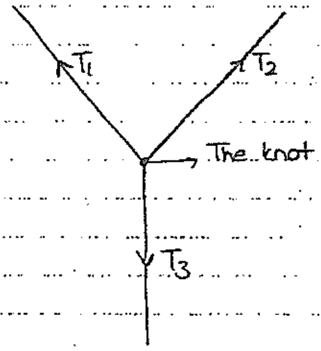
$T = F_g$

$T = mg$

Ex:

A traffic light connected by chain

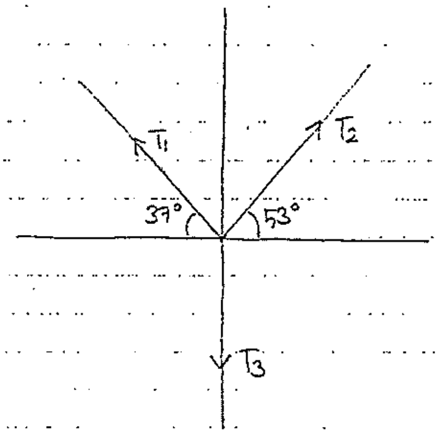




In equilibrium denge denklik $\Sigma\vec{F}=0$

$\Sigma F_x=0 \quad \Sigma F_y=0$

$\vec{a}=0 \quad a_x=0 \quad a_y=0$



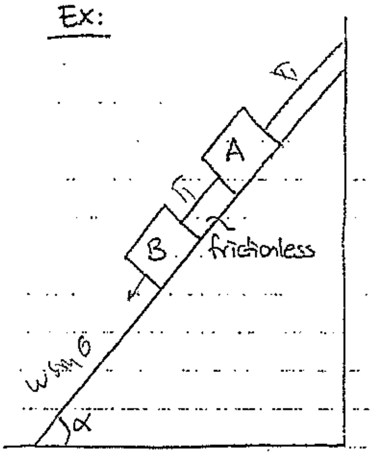
$\Sigma F_x=0$

$\boxed{T_2.\cos 53-T_1.\cos 37=0}$

$\Sigma F_y=0$

$\boxed{T_2.\sin 53+T_1.\sin 37-T_3=0}$

Ex:



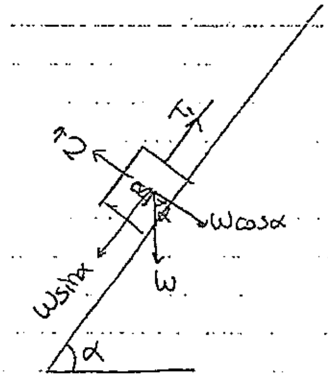
Two blocks.

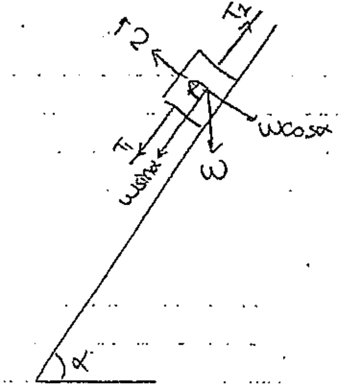
Each with weight w.

In terms of w and the angle α of the incline, calculate the tension in

a) the rope connecting the blocks

b) the rope that connects block A to the wall.





In equilibrium

$\Sigma\vec{F}=0 \quad \vec{a}=0$

$\Sigma F_x=0 \quad \Sigma F_y=0$

for B

$\Sigma F_x = 0 \quad T_1 - w\sin\alpha = 0$

$\boxed{T_1 = w\sin\alpha}$ → the rope connecting the blocks

$\Sigma F_y = 0 \quad N - w\cos\alpha = 0$

$\boxed{N = w\cos\alpha}$

for A

$\Sigma F_x = 0 \quad T_2 - T_1 - w\sin\alpha = 0$

$T_2 = T_1 + w\sin\alpha = \boxed{2w\sin\alpha}$ → the rope that connects block A to the wall

$\Sigma F_y = 0 \quad N - w\cos\alpha = 0$

$\boxed{N = w\cos\alpha}$

The Particle Under a Net Force:

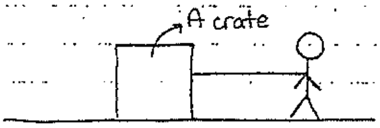
$\Sigma \vec{F} = m\vec{a}$

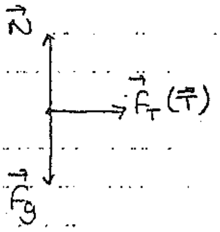
① Draw Free-Body diagram.

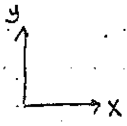
② $\Sigma F_x = ma_x$

$\Sigma F_y = ma_y$

Apply Newton's Second Law.







Apply Newton's Law:

$\Sigma \vec{F} = m\vec{a} \quad a_x = a$

$\Sigma F_x = ma_x$

$\Sigma F_x = ma$

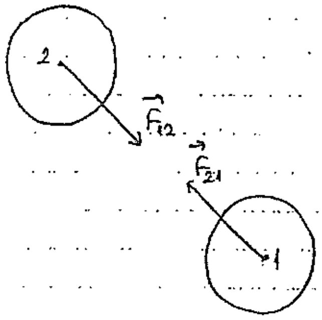
$\Sigma F_y = 0$

$\boxed{T = ma_x = ma}$

$N = w$ → the magnitude of $\vec{F}_g$

$N = mg$

Newton's Third Law:

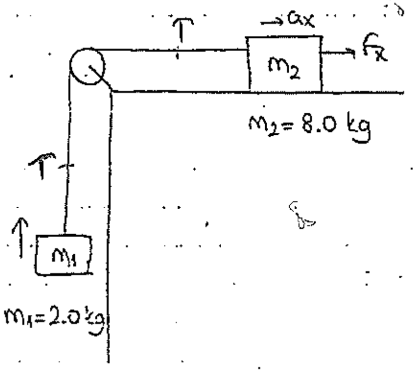


$\vec{F}_{12} = -\vec{F}_{21}$

☆ Action – Reaction Force

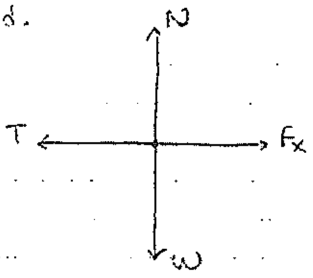
$\vec{F}_{AB}$ is the force exerted by A on B

Ex:



In the system shown in figure, a horizontal force F_x acts on the 8.0 kg mass.

a) For what values of F_x does the two kg mass accelerate upward.



T >20

G>40

m_2	m_1
$\Sigma \vec{F} = m\vec{a}$	$\Sigma F_y = m a_x$
$\Sigma F_x = m a_x$	$T - W = m a_x$
$\Sigma F_y = m a_y$	$T - m_1 g = m_1 a_x$
$a_y = 0$	
$\Sigma F_y = 0$	

$$\boxed{\begin{matrix} F_x - T = m_2 a_x \\ T - m_1 g = m_1 a_x \end{matrix}} \Rightarrow \begin{matrix} T - m_1 g = m_1 a \\ + \; F_x - T = m_2 a \\ \hline F_x - m_1 g = (m_1 + m_2) a \end{matrix}$$

$a_x = a$

$$\boxed{a = \frac{F_x - m_1 g}{m_1 + m_2}}$$

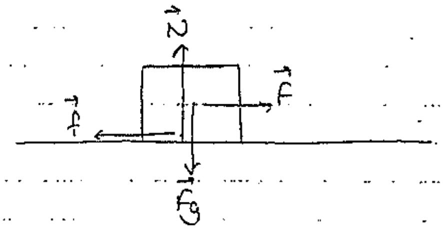
Answer: If $F_x > m_1 g$, the two kg mass will accelerate upward.

b) For what values of F_x is the tension in the cord is zero?

$$T = \frac{m_1}{m_1 + m_2}(F_x + m_2 g) \quad \text{If } F_x = -m_2 g$$

Forces of Friction:

When an object is in motion on a surface or through a viscous medium, there will a resistance to the motion. This is due to the interaction between the object and its vironment. The resistance is called the force of friction.



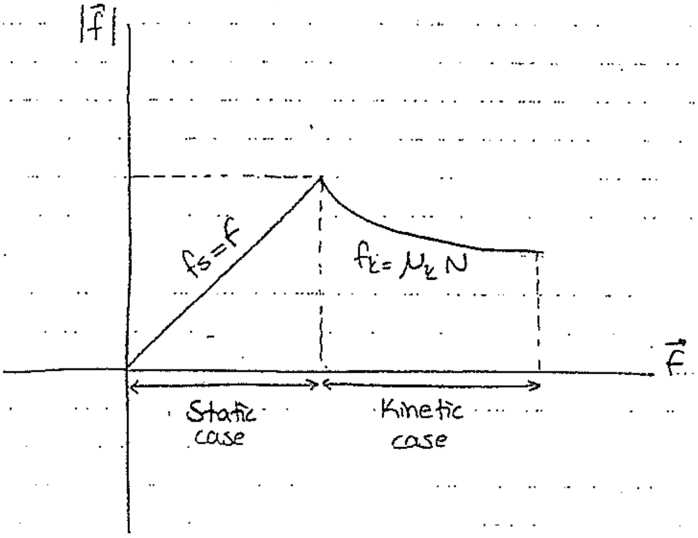
$f_s \rightarrow$ the static frictional force

$f_k \rightarrow$ the kinetic frictional force

$f_s = \mu_s . N$ → the coefficient of frictional force

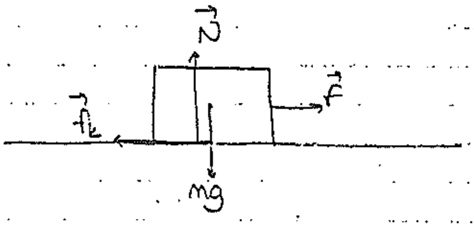
$f_k = \mu_k . N$ $\underline{f_s > f_k}$

★Static friction acts to keep the object from motion.



If $\vec{F}$ increases

$f_s \leq \mu_s N$



★Kinetic friction: The force of kinetic friction (f_k) acts when the object is in motion.

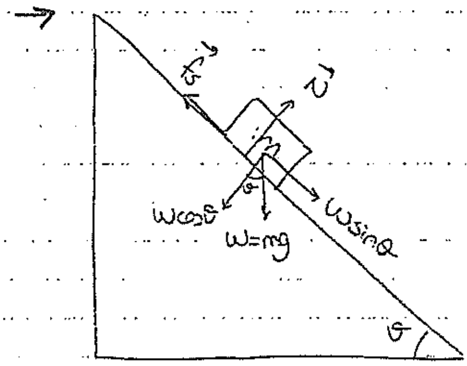
$\boxed{f_k = \mu_k . N}$

$\Sigma \vec{F} = m\vec{a}$

$\Sigma F_x = ma_x$

$\Sigma F_y = ma_y$

→



The block is sliding down the plane, so friction acts up the plane.

$\Sigma F_y = 0$ $\qquad$ $\Sigma F_x = ma_x$

$N - mg\cos\theta = 0$ $\qquad$ $a_x = a$

$mg\sin\theta - \mu . mg\cos\theta = 0$

$\sin\theta = \mu . \cos\theta$

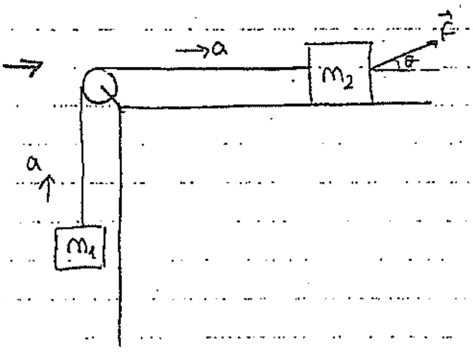
$f_s = \mu . N$

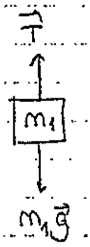
$\boxed{\mu = \tan\theta}$ → experimentally the value of coefficient of friction

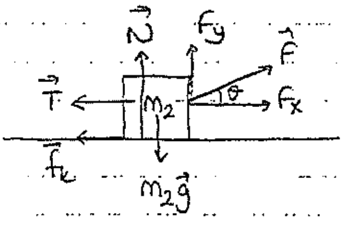
! For μ_s use the angle where the block just slips.

For μ_k use the angle where the block slides down with constant speed.

→







$\Sigma\vec{F} = m\vec{a}$

$\Sigma F_x = ma_x$

$\Sigma F_y = ma_y$

$\boxed{T - m_1 g = m_1 a}$

In y-direction

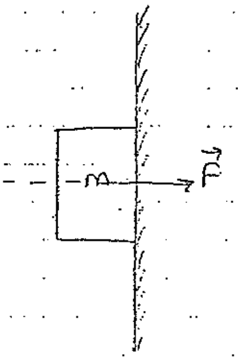
$\boxed{N - m_2 g + F_y = 0}$

In x-direction

$\boxed{F_x - T - f_k = m_2 a}$

$\boxed{f_k = \mu_k . N = \mu_k . (m_2 g - F_y)}$

Ex:



$\vec{F} = 12\,N$

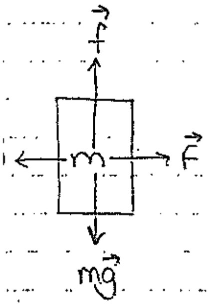
Weight of m = 5.0 N

$\mu_s = 0.60$

$\mu_k = 0.40$

Assume that initially the block is not moving.

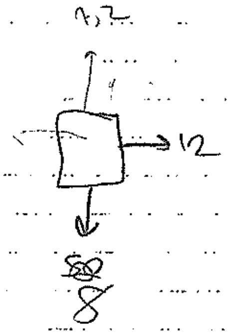
a) Will the block start moving?



$f_s = \mu_s . N$

$f_s < \mu_s . N$

$f_s > \mu_s . N$



$\Sigma \vec{F} = m.\vec{a}$

$\Sigma F_x = m.a_x$

$\Sigma F_y = m.a_y$

In x-direction:

$F - N = 0$

$N = F$

$N = 12\,N$

In y-direction:

$f - mg = 0$

$f = mg$

$f = 5.0\,N$

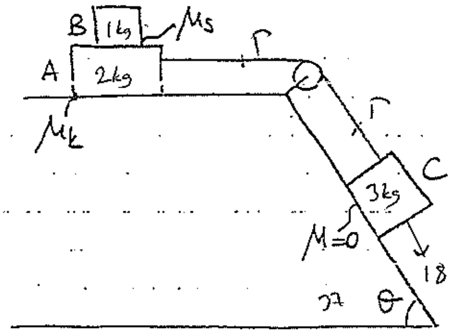
$f_s = (0.60).(12) = 7.2\,N$

Since $f < \mu_s . N$ The block does not slide.

b) In unit vector notation, what is the force exerted on the block by the wall?

$-12\hat{i} + 5\hat{j}\ N$

Ex:

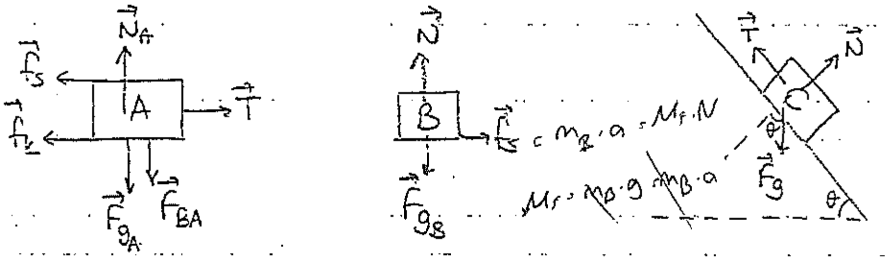


$w_A = 20N$ The system of blocks are released

$w_B = 10N$ from rest. A and B ⇒ move together.

$w_C = 30N$ $g = 10\ m/s^2$ $\theta = 37.0°$

a) Draw free body diagram for each block just after the release.



b) In terms of g and μ_s, what is the maximum acceleration that block B can have without sliding over block A?

$\vec{F}_{net} = m \cdot \vec{a}$ $\Sigma F_y = 0$ $f_s = m_B \cdot a$ $f_r ; ma$

$f_s = \mu_s \cdot N$ $N - m_B \cdot g = 0$ $\mu_s \cdot N = m_B \cdot a$ $\mu_r \cdot N = ma$

$N = m_B g$ $\mu_s \cdot m_B \cdot g = m_B \cdot a$

$$\boxed{a_{max} = \mu_s \cdot a}$$

c) If $\mu_k = 0.4$, what is the minimum μ_s between block A and block B so that B does not slip and (A and B) move together?

$a_A = a_B = a_C = a$

$\cancel{f_s} = m_B \cdot a$

$\cancel{T} - f_k - \cancel{f_s} = m_A \cdot a$

$m_C = g \sin\theta - \cancel{T} = m_C \cdot a$

$m_C = g \sin\theta - f_k = (m_A + m_B + m_C) \cdot a$

$f_k = \mu_k \cdot N$

$$\boxed{f_k = \mu_k \cdot (m_A + m_B) \cdot g}$$

$f_k = 12\ N$

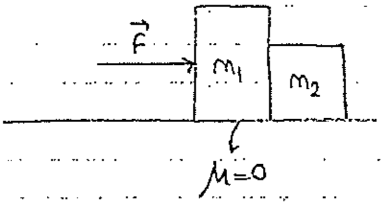
$a = 1\ m/s^2$

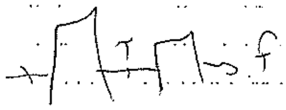
$f_s = \mu_s \cdot N$

$\mu_s = \frac{f_s}{N}$ $\quad \mu_s = \frac{m_B \cdot a}{N} = 0,1$ The minimum value

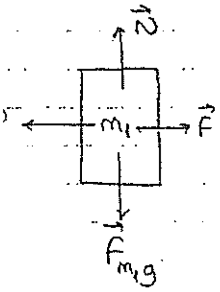
They move together if μ_s is equal or greater than this value.

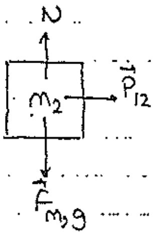
Ex:



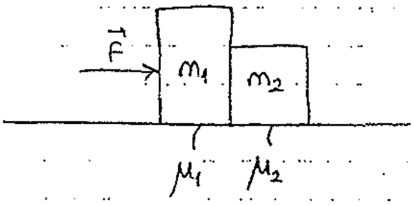


Draw free body diagram.

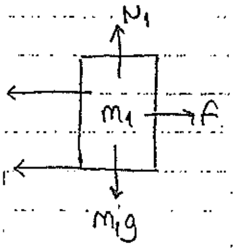


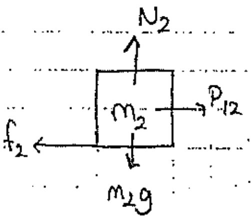


Ex:



Draw free body diagram.





$a_y = 0$

$\Sigma F_y = 0$

$N_1 - m_1 g = 0 \qquad N_2 - m_2 g = 0$

$N_1 = m_1 g \qquad N_2 = m_2 g$

Write Newton's second law in x-direction for each block.

The net force

$F - \cancel{P_{21}} - f_1 = m_1 a \qquad |\vec{P}_{21}| = |\vec{P}_{12}|$

$\cancel{P_{12}} - f_2 = m_2 a$

$$\boxed{F - f_1 - f_2 = (m_1 + m_2) a}$$

$$\boxed{a = \frac{F - \overbrace{\mu_1 \cdot m_1 \cdot g}^{f_1} - \overbrace{\mu_2 \cdot m_2 \cdot g}^{f_2}}{m_1 + m_2}}$$

$$P = \left(\frac{m_2}{m_1 + m_2}\right) \cdot \left[F - \cancel{(\mu_2 - \mu_1)} \cdot m_1 \cdot g\right] \quad \text{if } \mu_2 = \mu_1$$

#CHAPTER 6#

CIRCULAR MOTION

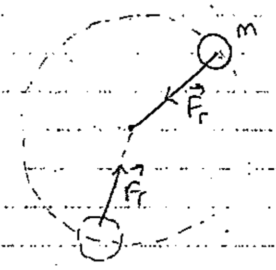
Newton's laws can be applied to other situations.

★ Objects traveling in circular paths.

$a_c = \frac{v^2}{r}$

a_c → directed toward the center of the circle.

The centripetal acceleration (a_c) is always perpendicular to the velocity.



Newton's Second Law

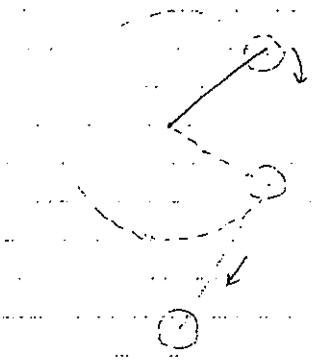
$\Sigma \vec{F} = m \vec{a}_c$

$$\boxed{\Sigma F = m \frac{v^2}{r}}$$

A force causing centripetal acceleration acts toward the center of the circle.

If the force vanishes, the object would move in a straight line path tangent to the rcle.

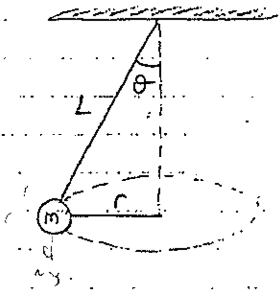
↳ to disappear suddenly
My keys have vanished

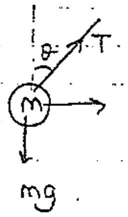


When the string breaks the puck moves in the direction tangent to the circle.

↳ to put things together

Conical Pendulum:





$\sum \vec{F} = m\vec{a}$

$T\cos\theta = mg$

$T\sin\theta = ma_c = m\frac{\vartheta^2}{r}$

$$\frac{T\sin\theta}{T\cos\theta} = \frac{m\frac{\vartheta^2}{r}}{mg}$$

$\tan\theta = \frac{\vartheta^2}{rg}$ $\quad \vartheta = \sqrt{\tan\theta . r . g}$

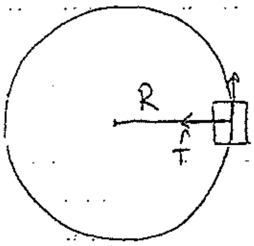
$\sin\theta = \frac{r}{L}$ $\quad r = L.\sin\theta$

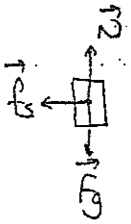
$\vartheta = \sqrt{r.g.\tan\theta} = \sqrt{L.g.\sin\theta.\tan\theta}$

ϑ is independent of m.

$T = \frac{2\pi r}{\vartheta}$ $\quad T = \frac{2\pi r}{\sqrt{Lg\sin\theta\tan\theta}} = 2\pi\sqrt{\frac{L^2.\sin^2\theta}{\cancel{L}g\cancel{\sin\theta}\tan\theta}} \Rightarrow T = 2\pi\sqrt{\frac{L.\sin\theta}{g.\tan\theta}}$

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}.\cos\theta}}$$





$v = ?$

$$\boxed{f_s = \mu_s \cdot N}$$

$$f_s = m \cdot a_c$$

$$f_s = m \cdot \frac{v^2}{r}$$

$$\boxed{v = \sqrt{\mu_s \cdot g \cdot r}}$$

Ex:

A 1500 kg car moving on a flat, horizontal road negotiates a curve whose radius is 35.0 m. If the coefficient of static friction between tires and dry pavement is 0.5, find the maximum speed the car can have in order to make the turn successfully.

$$f_{s_{max}} = \mu_s \cdot N$$

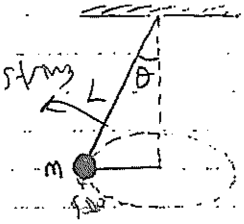
$$f_{s_{max}} = m \cdot \frac{v^2_{max}}{r} \qquad v_{max} = \sqrt{\frac{f_{s_{max}} \cdot r}{m}}$$

$\frac{1500 \cdot v^2}{35} = 1500 \cdot 0{,}5$

$$v_{max} = 13.1 \text{ m/s}$$

Problem 6/8:

Consider a conical pendulum.



$m = 80.0$ kg

$L = 10.0$ m

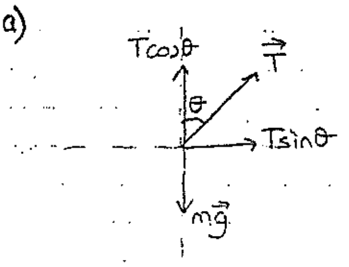
$\theta = 5.0°$ with vertical

Determine

a) the horizontal and vertical component of the force exerted by the string on the pendulum.

b) the radial acceleration of the bob.

a)



$$\vec{T} = \underset{\text{Tsin}\theta}{T_x}\hat{\imath} + \underset{\text{Tcos}\theta}{T_y}\hat{\jmath}$$

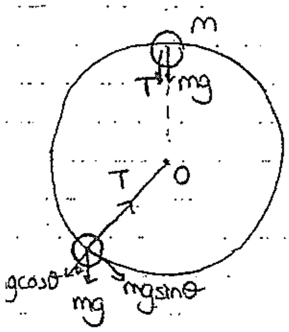
$$T\cos\theta = mg$$

$$\vec{T} = 686\hat{\imath} + 784\hat{\jmath}$$

b) $T\sin\theta = m a_c$

$$a_c = \frac{T\sin\theta}{m}$$

Non_uniform Circular motion:



→ Determine a_t (tangential acceleration) of the sphere.

$$\Sigma \vec{F} = m . \vec{a}$$

$$\Sigma F_r = T - mg\cos\theta = m\frac{v^2}{r}$$

$$\Sigma F_t = mg\sin\theta = m a_t$$

$$\boxed{a_t = g\sin\theta}$$

→ Find the tension in the cord at any instant when the speed of the sphere v and

cord makes an angle θ with vertical.

$$T = mg\cos\theta + m\frac{v^2}{r}$$

$$\boxed{T = mg\left(\frac{v^2}{rg} + \cos\theta\right)}$$

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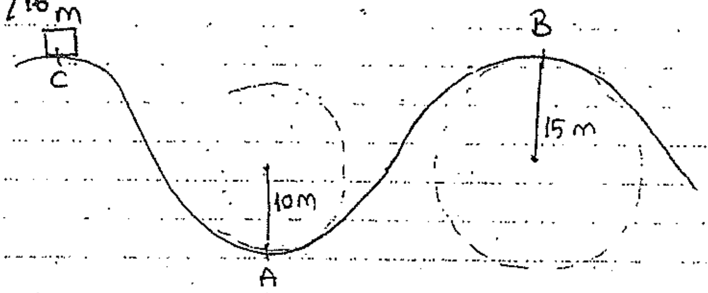


$m = 500$ kg

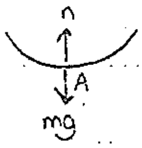
$v_A = 20.0$ m/s

$$5000 + \cancel{5000}\ \frac{500 \cdot 400}{10}$$

$$25000$$

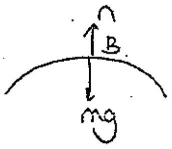
What is the force exerted by the track on the car at point A?

$$\Sigma F_r = m\frac{v^2}{R_1}$$



$$\Sigma F_r = n - mg = m\frac{v^2}{R_1}$$

$$\boxed{n = mg + m\frac{v^2}{R_1}}$$



$$\Sigma F_r = n - mg = -m\frac{v^2}{R_2}$$

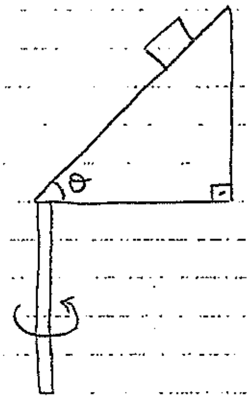
when $n=0$

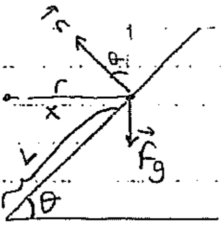
$$v=\sqrt{gR_2}$$

$$T = mg\cos\theta + m\frac{v^2}{r}$$

$$T = mg\left(\frac{v^2}{rg} + \cos\theta\right)$$

6/61 6/42





$r = L\cos\theta$

$\Sigma F_x = ma_x$

$n\sin\theta = m\frac{v^2}{r}$

$\Sigma F_y = ma_y = 0$

$$\left(\begin{array}{l} n\sin\theta = m\frac{v^2}{r} \\ n\cos\theta = mg \end{array}\right.$$

dividing

$\tan\theta = \frac{v^2}{rg}$ $\qquad v = \sqrt{r.g.\tan\theta}$

$$v = \sqrt{r.\sin\theta.g.\frac{\sin\theta}{\cos\theta}}$$

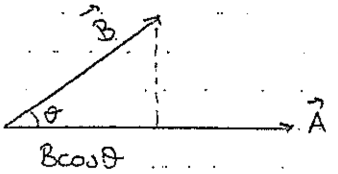
$$\boxed{v = \sqrt{g.L.\sin\theta}}$$

CHAPTER 7

$W = \vec{F} \cdot \Delta \vec{r}$

Dot product or scalor product

$\vec{A}, \vec{B}$



$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \cos\theta$

$= A \cdot B \cdot \cos\theta$

$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k})(B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$

$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{i} \cdot \hat{k} = 0$

$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$

$\boxed{\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| \cdot |\vec{B}|}}$ $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ $|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$

$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ (Commutative law)

$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$ (Distributive law of multiplication)

If $\underbrace{\vec{A} \parallel \vec{B}}_{\text{parallel}}$, $\theta = 0^\circ$ $\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}|$

If $\vec{A} \perp \vec{B}$, $\theta = 90^\circ$ $\vec{A} \cdot \vec{B} = 0$

$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$

If $\vec{A} = \vec{B}$, $\vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2$

Forces:

1) Constant forces ($\vec{F}$ = constant. The simplest situation)

2) Position dependent forces ($\vec{F}(\vec{r})$, $F(x)$)

→ Spring force $F = -kx$

→ Gravitational force $F = G \frac{m_1 m_2}{r^2}$

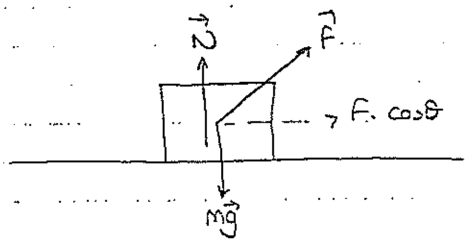
→ Coulomb force $k \frac{\text{[illegible]}}{r^2}$

3) Velocity dependent forces ($\vec{F}(\vartheta) \rightarrow$ Damping forces $\vec{F} = -b\vartheta^{\wedge}$

If the particle is moving in a resistance medium)

4) Time dependent forces ($F(t) = F_0 \sin\omega t$)

(The sinusoidal force)



$\boxed{W = F.\Delta r.\cos\theta}$

The unit of work is joule. 1 joule = 1 Newton x 1 meter

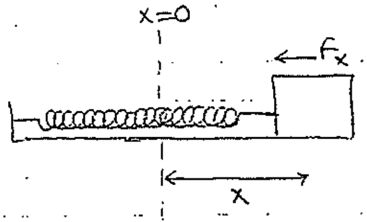
W+ → energy is transferred to the system

W− → energy is transferred from the system

$$W = \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x \, dx$$

$$\Sigma W = W_{ext} = \int_{x_i}^{x_f} (\Sigma F_x) dx$$

$$\Sigma W = W_{ext} = \sum_{forces} (\vec{F}.d\vec{r})$$



The force exerted by the spring $\boxed{F_s = -kx}$

$\boxed{\vec{F}_s = F_x\hat{i} = -kx\hat{i}}$

Work done by a spring:

$$W_s = \int \vec{F}_s . d\vec{r} = \int_{x_i}^{x_f} (-kx\hat{i})(dx\hat{i}) = \int_{-x_{max}}^{0} (-kx)dx = \boxed{\frac{1}{2}kx_{max}^2}$$

The net work done as the block moves from $-x_{max}$ to x_{max} is zero.

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\boxed{W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2}$

$\vec{F} \rightarrow$ constant

$\vec{F} \cdot \Delta\vec{r}$

$W = F \cdot \Delta r \cdot \cos\theta$

If $\vec{F} \perp \Delta\vec{r}$ No work is done.

Sign of work depends on the direction of F relative to Δr.

If $\vec{F}$ is in the same direction then $\theta = 0$ $W = F \cdot \Delta r$

* Work is a <u>scalar</u> quantity.

Unit of force in SI is 1N

$$W \rightarrow \underbrace{1N.m}_{1J}$$

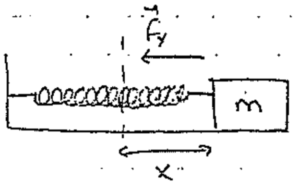
$1J = 10^7$ erg

Work done by a variable force

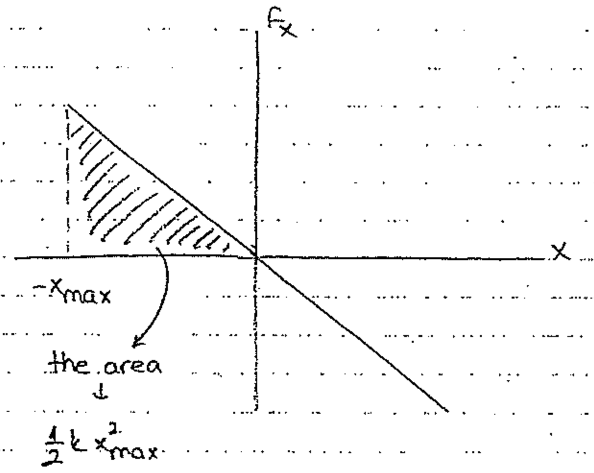
$$W = \int \vec{F}(r) \cdot d\vec{r}$$

$$W = \int_{x_i}^{x_f} F(x) dx$$

Work done by a spring:



$$\boxed{W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2}$$



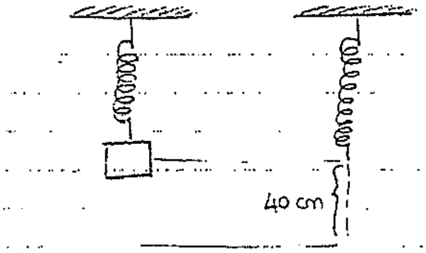
Kinetic Energy:

$K = \frac{1}{2}m\vartheta^2$

$W = \Delta K$

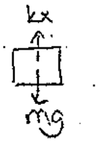
Ex:

An ideal spring hung vertically from ceiling when 2.0 kg mass hang at rest from it, the spring is extended 4.0 cm. A downward external force is now applied to the mass to extend the spring and additional 8.0 cm. While the spring is being extended by the force, what will be the work done by the spring?



$W_s = \int_{y_i}^{y_f} F_s dy$ $F_s = -ky$ $y_i = 4\times10^{-2}$ m

$y_f = 12\times10^{-2}$ m



$kx = mg$

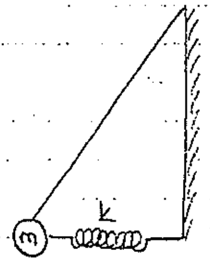
$k = \frac{mg}{x}$

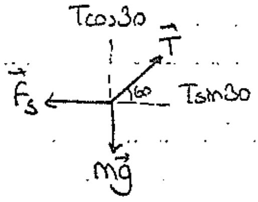
$$\boxed{W_s = \frac{1}{2}ky_i^2 - \frac{1}{2}ky_f^2}$$

$W_s = -32\ J$ //

Ex:

The figure shows a steel ball of mass m hanging on a string on a vertical wall. The ball pushes on a spring of force constant k and is at rest. Find the stored energy in the spring.





$T\sin30 - f_s = 0$

$T\cos30 - mg = 0$

$$\boxed{\begin{matrix} T\sin30 - kx = 0 \\ T\cos30 - mg = 0 \end{matrix}}$$

$T\sin30 = kx$
$T\cos30 = mg$

$x = \frac{mg}{k}\tan30$ //

Ex:

$\vec{A} = 3\hat{i} + 4\hat{j}$

$\vec{B} = 2\hat{i} - 2\hat{j}$

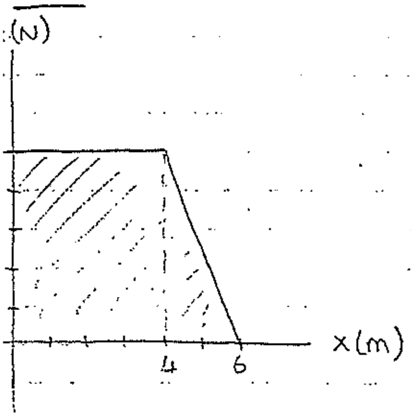
$\vec{A}\cdot\vec{B} = |\vec{A}|\cdot|\vec{B}|\cdot\cos\theta$

$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$ $\quad |\vec{A}| = \sqrt{A_x^2 + A_y^2} \quad |\vec{B}| = \sqrt{B_x^2 + B_y^2}$

$\vec{A}\cdot\vec{B} = (3\hat{i} + 4\hat{j})(2\hat{i} - 2\hat{j})$

$(\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y)$

Ex:



$\left.\begin{array}{l} x_i = 0 \\ x_f = 6 \end{array}\right\}$ find the w_{F_x} (work done by F_x)

$w_{F_x} = \text{area} = 25\,N$ //

Ex:

If a force $\vec{F} = 6\hat{i} - 2\hat{j}$ (N) acts on a particle that undergoes a displacement $\Delta\vec{r} = 3\hat{i} + \hat{j}$ (m)

Find;

a) the work done by the force on the particle

b) the angle between $\vec{F}$ and $\Delta\vec{r}$

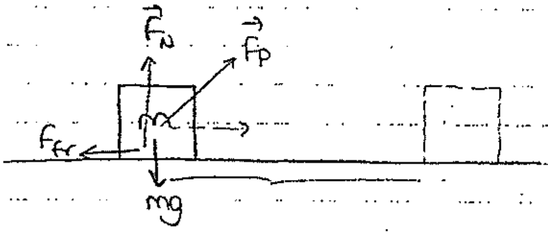
a) $w = \vec{F}\cdot\Delta\vec{r}$

$= (6\hat{i} - 2\hat{j}).(3\hat{i} + \hat{j})$

$= 18 - 2 = 16$ joule //

b) $\cos\theta = \frac{\vec{F}\cdot\Delta\vec{r}}{|\vec{F}|.|\Delta\vec{r}|}$

The net work done can be calculated two equivalent way:



1) The net work done on an object is the algebraic sum of the work done by each force,

ce. work is a scalar quantity.

$$\boxed{W_{net} = w_p + W_N + w_{f_{fr}} + w_{f_g}}$$

2) The net work can also be calculated by first determining the net force on the object, then taking its component along the displacement

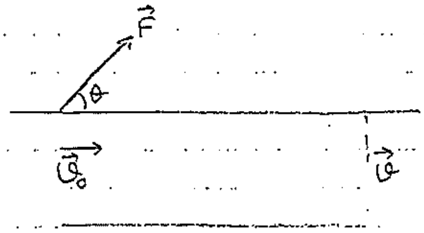
$(F_{net})_x = F_p\cos\theta - F_{fr}$

$$\boxed{W_{net} = (F_p\cos\theta - F_{fr})\,x}$$

In y-direction there is no displacement and no work.

Work (W) is energy transferred to or from an object by means of a force acting on the object. Energy transferred to the object is positive and energy transferred from the object is negative work.

Work-Kinetic Energy Theorem:



$\vec{F}$ accelerates the particle

$\vec{v}_0$ initial velocity

$\vec{v}$ final velocity

$\vec{F}$ is constant $\boxed{F_x = m.a_x}$.

$v^2 - v_0^2 = 2a_x d$

$a_x = \frac{1}{2d}\,(v^2 - v_0^2)$

$F_x = m\left(\frac{1}{2d}(v^2 - v_0^2)\right)$

$\underline{F_x d} = \underline{\frac{1}{2}mv^2} - \underline{\frac{1}{2}mv_0^2}$

$W = \underbrace{K_f - K_i}_{\Delta K}$

$\boxed{W = \Delta K}$ → work-kinetic energy theorem

Kinetic Energy:

K (J)

(The ability to do work)

$K = \frac{1}{2}mv^2$

$W = \Delta K$

$\boxed{K_f = K_i + W}$

$1\ J = 1\ N.m$
$= 1\ kg.m/s^2$

$1\ eV = 1.6 \times 10^{-19}\ J$

$1\ keV = 10^3\ eV$

$1\ MeV = 10^6\ eV$

$1\ GeV = 10^9\ eV$

If $\vec{F}$ is variable

$$W_{net} = \int_{x_i}^{x_f} (\Sigma F_x)dx$$

$$W_{net} = \int_{x_i}^{x_f} m a_x \, dx$$

$$a_x = \frac{d\vartheta_x}{dt} = \int_{x_i}^{x_f} m \cdot \frac{d\vartheta_x}{dt} dx \; \vartheta_f$$

$$W_{net} = m \int_{x_i}^{x_f} \vartheta d\vartheta$$

$$W_{net} = m \frac{\vartheta^2}{2} \Big|_{\vartheta_i}^{\vartheta_f}$$

$$W_{net} = \underbrace{\frac{1}{2} m \vartheta_f^2 - \frac{1}{2} m \vartheta_i^2}_{\Delta K}$$

The particle moves along an arbitrary curved path in three dimension.

$$W = \int \vec{F} \cdot d\vec{r}$$

$$\vec{F} = F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k}$$

$$d\vec{r} = dx \hat{\imath} + dy \hat{\jmath} + dz \hat{k}$$

$$W = \int_i^f (F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k})(dx \hat{\imath} + dy \hat{\jmath} + dz \hat{k})$$

$$W = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz$$

Ex:

$\vec{F} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ acts on a particle as it moves from the point $(\hat{\imath} + 2\hat{\jmath} - \hat{k})$ m. to the nt $(-2\hat{\jmath})$ m. If the initial $K_E = 12$ J, find its final K_E, in J.

$$W = \int_i^f (F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k})(dx \hat{\imath} + dy \hat{\jmath} + dz \hat{k})$$

$$W = \int_{1}^{0} \underbrace{F_x}_{1} dx + \int_{2}^{-1} \underbrace{F_y}_{2} dy + \int_{-1}^{0} \underbrace{F_z}_{-1} dz$$

$W = -10\ J$

$W = \Delta K \Rightarrow W = K_f - K_i$

$-10 = K_f - 12$

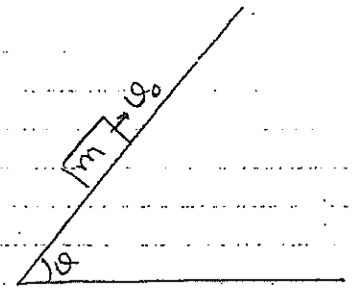
$K_f = 2\ J$ //

Ex:

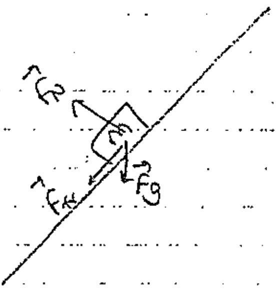
A block of mass $m = 2.0$ kg

The coefficient of kinetic friction $\mu_k = 0.25$

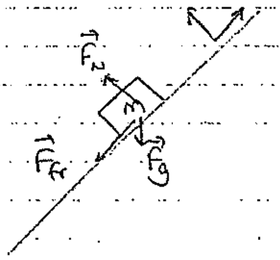
a) Draw free body diagram of the block and show all the forces acting on it.



$\theta = 37.0°$



b) Calculate the work done on the block by each force as a function of the distance d traveled up the plane.



$W_N = 0$

$f_k = \mu_k \cdot mg\cos\theta$

$f_{fr} = \mu_k \cdot N$

$\Sigma F_y = 0$

$N - mg\cos\theta = 0$

$N = mg\cos\theta$

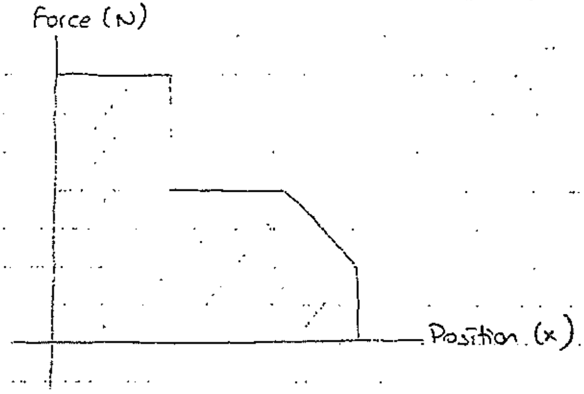
$W_{F_{fr}} = \vec{F}_{fr} \cdot \vec{d}$

$W_{F_{fr}} = |\vec{F}_{fr}| \cdot |\vec{d}| \cdot \cos 180°$

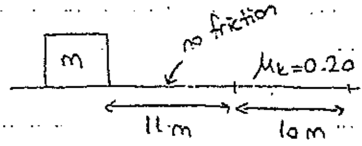
$W_{F_{fr}} = -f_{fr} \cdot d$

$= -4.0 d$ //

c) How far up the plane the block travels before it stops momentarily?



Ex:



A crate starting from rest, is pulled across a floor with constant horizontal force of 225 N. what is the final speed of the crate after being there 21.0 m?

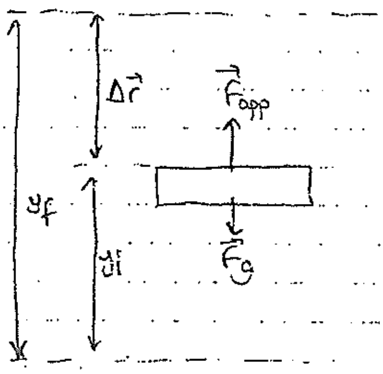
m = 66 kg

?

$w_{net} = \Delta K = K_f - K_i$

Potential Energy:

Potential energy is energy determined by the configuration of a system in which the components of the system interact by forces.



$\Delta y = y_f - y_i$

The work done by agent on the book_Earth system

$\downarrow$

$(mgy_f - mgy_i)$

$\boxed{W_{ext} = \vec{F}_{app} \cdot \Delta \vec{r}}$

Gravitational potential energy is the energy associated with an object at a given location above the surface of the earth.

$$W_{ext} = \vec{F}_{app} \cdot \Delta\vec{r}$$

$$\Delta\vec{r} = (y_f - y_i)\hat{j}$$

$$W_{ext} = (mg\hat{j})\left[(y_f - y_i)\hat{j}\right] \Rightarrow \boxed{W_{ext} = \underbrace{mgy_f}_{U_{y_f}} - \underbrace{mgy_i}_{U_{y_i}}}$$

$$\boxed{W_{ext} = U_{y_f} - U_{y_i}}$$

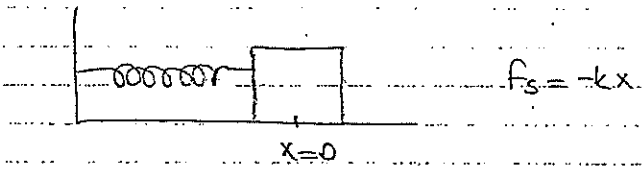
mgy → identified as the gravitational potential energy

↳ $U_y = mgy$

Work may change the gravitational potential energy.

$$\boxed{W_{ext} = \Delta U_g}$$

→ Elastic Potential Energy:



Elastic potential energy is associated with a spring.

The force the spring exerts (on a block) is $f_s = -kx$

The work done by F_{app}

$$W_{app} = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2$$

$$W = \int_{x_i}^{x_f} (-kx)\,dx$$

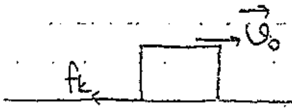
$$-k\frac{x^2}{2}\Big|_{x_i}^{x_f}$$

$$-\frac{1}{2}kx_f^2 + \frac{1}{2}kx_i^2 \Rightarrow \boxed{W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2}$$

The elastic potential energy can be thought of as the energy stored in the deformed ring.

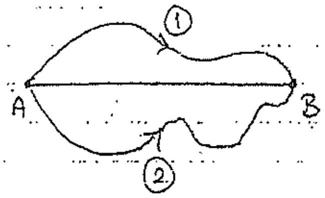
Internal Energy:

The energy is associated with an object's temperature is called the internal ergy. (E_{int})



The friction does work and increases the internal energy of the surface.

Conservative Forces:



If $\vec{F} \rightarrow$ conservative $w_{AB①} = w_{AB②}$

The work done by a conservative force on a particle moving on any closed path zero.

The work done by a conservative force on a particle moving between any two nts is independent of the path taken by the particle.

→We can associate a potential energy for a system with any conservative force acting ween members of the system.

This can be done only for conservative forces.

In general, $\boxed{w_{int} = -\Delta U}$

$\boxed{w_{ext} = \Delta U}$

★Positive work done by an outside agent on a system causes an increase in e potential energy of the system.

$\Delta K + \Delta U = 0 \quad \boxed{\Delta K = -\Delta U}$

Work done on a component of a system by a conservative force internal to an isolated system causes a decrease in the potential energy of the system.

$$\left.\begin{array}{l} F_s \text{ (spring force)} \\ F_g \text{ (gravitational force)} \end{array}\right\} \text{conservative forces}$$

$$\boxed{E_{mech} = K + U}$$

mechanical energy of the system

Conservative Forces and Potential Energy:

Define the potential energy function, U, such that the work done by a conservative force equals the decrease in the potential energy of the system.

$$W_{int} = \int_{x_i}^{x_f} F_x\, dx = -\Delta U$$

ΔU is negative when F and x are in the same direction.

$$\boxed{F_x = -\frac{dU}{dx}}$$

The conservative force is related to the potential energy function.

$U_g = mgy$ — F_g

$U_s = \frac{1}{2}kx^2$ — F_s

Conservative forces and potential energy check:

$$F_s = -\frac{dU}{dx} = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right)$$

$$F_s = -kx \quad \left(\frac{1}{2}(2.k.x)\right)$$

$$F_g = -\frac{dU_g}{dy} = -\frac{d}{dy}(mgy)$$

$$\boxed{F_g = mg}$$

Conservation of Energy:

$$\boxed{E_{mech} = K + U}$$

For isolated systems $\Delta E_{mech} = 0$

$$\Delta K + \Delta U = 0$$

$\Delta K = K_f - K_i$

$\Delta U = U_f - U_i$

$$(K_f - K_i) + (U_f - U_i) = 0$$

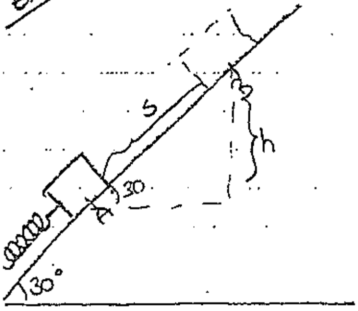
$$\boxed{K_f + U_f = K_i + U_i}$$

The conservation of energy for isolated systems with no nonconservative forces acting.

Conservation of energy becomes

$$\boxed{\Delta E_{sys} = 0}$$

Ex:



A 2.0 kg block is placed against a spring on a frictionless incline. $k = 19.6$ N/cm (spring constant)

The spring is compressed 20 cm and then released.

How far along the incline does it send the block?

$E_{mech} = K + U$

$\Delta K + \Delta U = 0$

$$\cancel{(K_f)}_B + (K_i)_B = \cancel{(K_i)}_A + (U_i)_A$$

$$\left(\frac{1}{2}kx^2\right)_A = mgh \qquad h = s.\sin 30$$

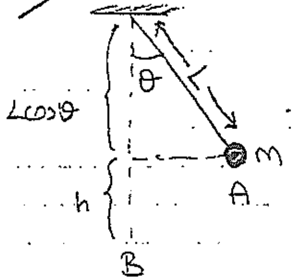
$$\frac{1}{2}.k.x^2 = m.g.s.\sin 30$$

$$\boxed{s = \frac{\frac{1}{2}k.x^2}{m.g.\sin 30}} \qquad S = 4\text{ m}$$

What is the elastic potential energy of the compressed spring?

$$U_s = \frac{1}{2}kx^2 = 39.25$$

EX:-



A thin rod
It mass is negligible.

$\theta = 30.0°$

$L = 2.0\ m$

How fast is the ball moving at its lowest point?

$$\Delta K + \Delta U = 0$$

$$\cancel{K_A} + U_A = K_B + \cancel{U_B}$$

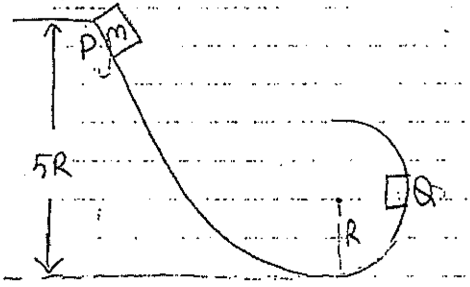
$$mgh = \frac{1}{2}mv_B^2$$

$$(h = L(1-\cos\theta))$$

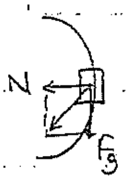
$$\cancel{m}gL(1-\cos\theta) = \frac{1}{2}\cancel{m}v_B^2$$

$$v_B = \sqrt{2.g.L.(1-\cos\theta)} //$$

EX:-



It is released from rest at point P. What is the net force acting on it at point Q?



$$F_{net} = \sqrt{(mg)^2 + N^2} \qquad N = \frac{mv_q^2}{R}$$

$$\boxed{F_{net} = \sqrt{(mg)^2 + \left(\frac{mv_q^2}{R}\right)^2}}$$

$$\cancel{K_p} + U_p = K_q + U_q$$

$$m.g(5R) = \frac{1}{2}m.(v_q)^2 + m.g.R$$

find v_q

$$F_{net} = \sqrt{65\ mg} //$$

$$mv_q^2 = 8mgR$$

Force and Potential Energy:

$$\boxed{F_x = -\frac{dU(x)}{dx}}$$

$$F_y = -\frac{dU(y)}{dy}$$

$$F_z = -\frac{dU(z)}{dz}$$

$$F_x = -\frac{\Delta U(x)}{\Delta x} \qquad F_y = \frac{-\Delta U(y)}{\Delta y} \qquad F_z = \frac{-\Delta U(z)}{\Delta z}$$

$$\Delta x \to 0 \qquad \Delta y \to 0 \qquad \Delta z \to 0$$

$$F_x = -\frac{\partial u}{\partial x}$$

$$F_y = -\frac{\partial u}{\partial y}$$

$$F_z = -\frac{\partial u}{\partial z}$$

$$\vec{F} = -\left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right)$$

$\boxed{\vec{F} = -\vec{\nabla} u}$ → This operation is called the gradient of u.

$\vec{F} = -\vec{\nabla}(mgy) \qquad u = mgy$

$$\vec{F} = -\left(\frac{\partial(mgy)}{\partial x}\hat{i} + \frac{\partial(mgy)}{\partial y}\hat{j} + \frac{\partial(mgy)}{\partial z}\hat{k}\right)$$

$$\boxed{\vec{F} = -(mg)\hat{j}}$$

For the potential energy

$u = 3x^2 + 2xy + 4y^2z$

$\vec{F} = ?$

$$F_x = -\frac{\partial u}{\partial x} = -(6x+2y)$$

$$F_y = -\frac{\partial u}{\partial y} = -(2x+8yz)$$

$$F_z = -\frac{\partial u}{\partial z} = -(4y^2)$$

$$\vec{F} = -(6x+2y)\hat{i} - (2x+8yz)\hat{j} - (4y^2)\hat{k}$$

Ex: $F = F_0 + Cx$

$F_0 = 5.0\ N$

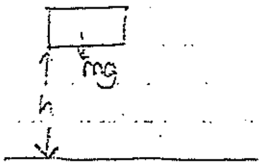
$C = -2.0\ N/m$ Initially it is at rest and its position is $x=0$

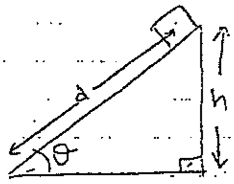
$m = 100\ g$

a) Calculate the work done by the force when the point $x = 8.0\ m$

b) Calculate $W_1 = ?$ $W_2 = ?$ Decide whether this force is conservative or not.

$W_1 + W_2 = 0$ conservative (path independent)





$W = mgh$ $W = mg\underbrace{(\sin\theta d)}_{h}$ $\left(\sin\theta = \frac{h}{d} \Rightarrow h = \sin\theta d\right)$

$K_i + \Sigma U_i = K_f + \Sigma U_f$

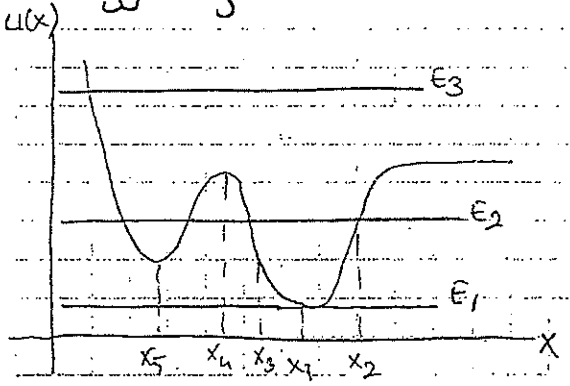
$\Delta K + \Delta U = 0$

$$f_x = -\frac{\Delta U}{\Delta x} \Rightarrow \boxed{f_x = -\frac{dU}{dx}}$$

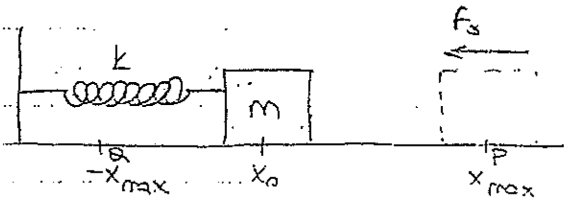
→ The total mechanic energy of a system remains constant if the force is conservative.

→ The work done by a nonconservative force can not be represented by a potential energy function.

Energy diagrams

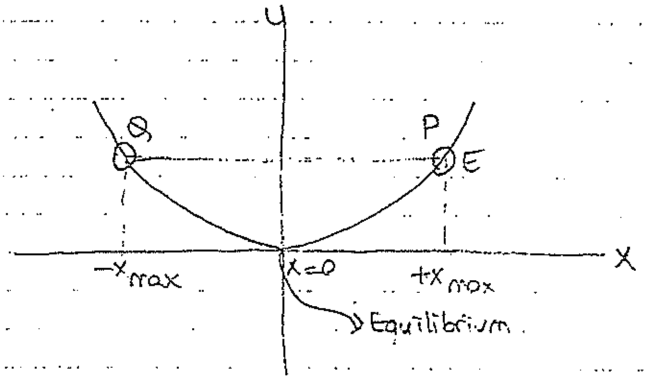


$x_1, x_5 \rightarrow$ min



$U_s = \frac{1}{2}kx^2$

$E = K + U$

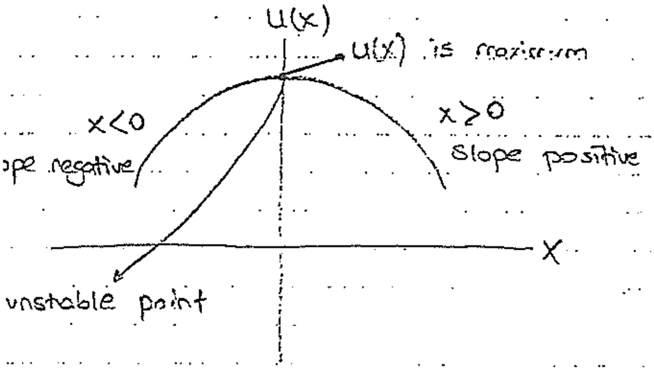


P and Q are turning points.

$$F_x = -\frac{dU(x)}{dx}$$

In the spring-mass system

$\pm x_{max} \rightarrow$ Turning points



(unstable point)

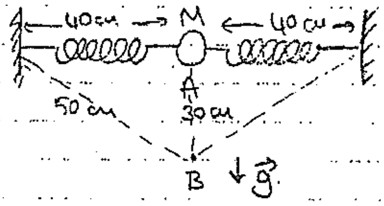
$E = \frac{1}{2}m\vartheta^2 + U(x)$

$E - U(x) = \frac{1}{2}m\vartheta^2 \Rightarrow E - U(x) = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$

$\vartheta = \frac{dx}{dt}$

$$\sqrt{\frac{2(E-U(x))}{m}} = \frac{dx}{dt}$$

Example:



Two identical springs

Equilibrium length 40 cm

k=600 N/m

a) What is the total spring potential energy when the mass is at point B?

$U = \frac{1}{2}kx^2$

$50-40 = 10\ cm = 0{,}1\ m$

$U_{tot} = ?$

$U_{tot} = 2\cdot\left(\frac{1}{2}\cdot 600\cdot(0{,}1)^2\right) = 6J$

b) what is the mass M?

$E_i = E_f$

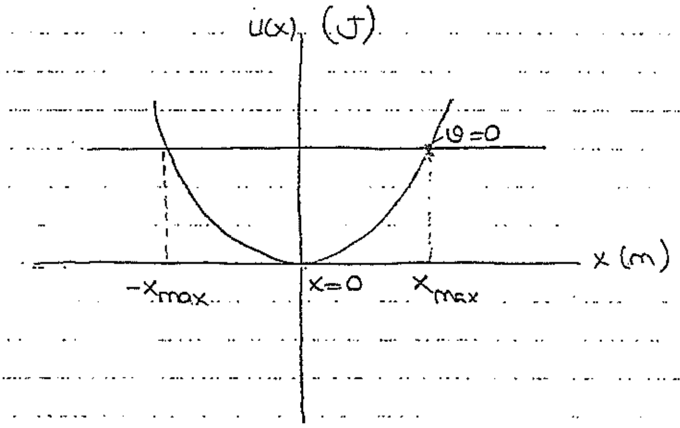
$$K_i + \underbrace{U_i}_{U_g+U} = K_f + \underbrace{U_f}_{U_g+U}$$

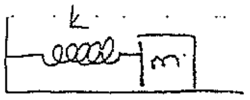
$0+0+0 = 0+(Mg(-h)) + \frac{1}{2}\cdot k\cdot x^2$

$0 = -Mgh + \frac{1}{2}kx^2$

$\frac{1}{2}kx^2 = Mgh$

$$\boxed{M = \frac{\frac{1}{2}kx^2}{gh}}$$





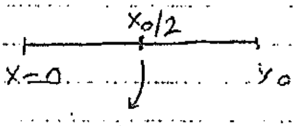
$m = 0.5\ kg$

$k = 100\ N/m$

$x_0 = 1.0\ m$

$E = ?$

$E = U(x) = \frac{1}{2}kx^2$



$F_x = -\frac{dU(x)}{dx}$

It has both kinetic and potential energy.

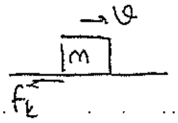
$E = K + U \qquad K = E - U(x)$

$\downarrow$

$\frac{1}{2}k\left(\frac{x_0}{2}\right)^2 \qquad \vec{a} = \frac{F}{m}$

Conservation of Energy and Friction:

If we have friction then the total energy reduces and transforms into heat, as thermal ergy due to friction between floor and block.



$\Delta K + \Delta U + \Delta E_{int} = 0$ (dissipated energy due to friction) = internal energy

One can write the equation as A_initial state, B_final state.

$K_A + U_A = E_A$ $\qquad \Delta K = K_B - K_A$ $\qquad K_A + U_A - W_{fr} = K_B + U_B$

$K_B + U_B = E_B$ $\qquad \Delta U = U_B - U_A$ $\qquad \Delta K + \Delta U = -f_{fr} \cdot d$

Ex/ Potential energy of a 0.2 kg particle moving along x_axis is given by $U(x) = 8x^2 - 2x^4$ (J) hen the particle is at $x = 1$ m $a = ?$

$$F_x = -\frac{dU(x)}{dx} = -16x + 8x^3 = 8 - 16 = \frac{-8}{0.2} = 0.2 \cdot a \quad a = -40 \text{ m/s}^2 //$$

$K_f + U_f = K_i + U_i$

$\Delta K + \Delta U = -f_k d$

In general, if friction is acting in a system,

$$\boxed{\Delta K = \Sigma W_{\substack{other \\ forces}} - f_k d}$$

A friction force transforms kinetic energy in a system to internal energy.

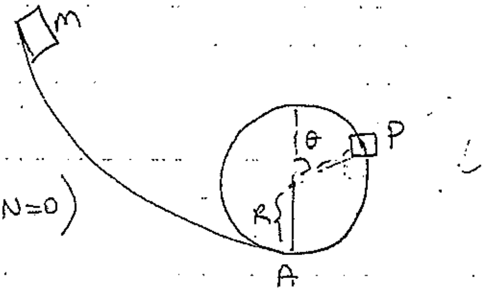
$\Delta E_{int} = f_k d$

If nonconservative forces act within a nonisolated system and the external influence on the stem is by means work

$$\boxed{\Delta E_{mech} = -f_k d + \Sigma_{\substack{other \\ forces}}}$$

Ex: A cart sliding down the smooth track shown enters the bottom of the circular portion with a speed $2\sqrt{gR}$. The cart looses contact with the track at point P. $\rangle$ (N=0)

Find the value of cosθ.



Use conservation of energy.

$$K_A + \underset{0}{\cancel{U_A}} = K_P + U_P$$

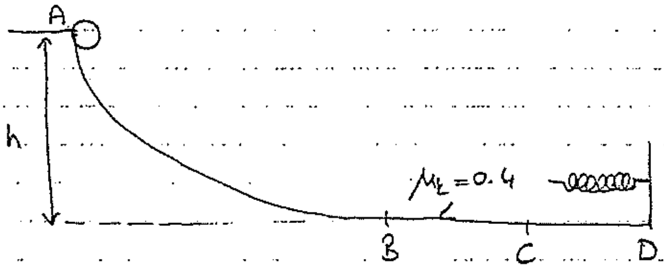
$$\boxed{\frac{1}{2} m(\vartheta_A)^2 = \frac{1}{2} m(\vartheta_P)^2 + m.g.R(1+\cos\theta)}$$

$$\boxed{mg\cos\theta - \underset{0}{\cancel{N}} = m\frac{\vartheta_P^2}{R}}$$

$\cos\theta = \frac{2}{3}$ //

$\Delta E_{mech} = 0$ //

Ex: A 2.0 kg block is released from point A.



AB and CD are frictionless.

$k = 2000\ N/m^2$

$(g = 10\ m/s^2)$

a) If the speed of the block is 10 m/s at point B, what is the height h of the point A?

$E = K + U$

$\cancel{K_A} + U_A = K_B + \cancel{U_B}$ $(E_A = E_B)$

$\cancel{m} g h = \frac{1}{2} \cancel{m}(\vartheta_B)^2$

$h = \frac{(\vartheta_B)^2}{2g} = \frac{100}{20} = 5\ m$ //

b) If the kinetic energy of the block at point C is 40 J, what is the distance between

and C?

$W = \Delta K$

$-\mu_k \cdot m \cdot g \cdot d = K_c - K_B$

$-\mu_k \cdot m \cdot g \cdot d = \underbrace{\frac{1}{2} \cdot m \cdot (v_c)^2}_{K_c} - \frac{1}{2} \cdot m (v_B)^2$

$\mu_k = 0.4$ $v_B = 10$ m/s $K_c = 40$ J $g = 10$ m/s^2

m = 2.0 kg

d = 7.5 m //

c) What is the distance that the block compresses the spring from its equilibrium

sition before coming to rest?

$\frac{1}{2} \cdot m \cdot (v_c)^2 = \frac{1}{2} \cdot k \cdot x^2$

$40 = \frac{1}{2} \cdot \overset{1000}{\cancel{2000}} \cdot x^2$

$x^2 = 0.04$

$x = 0.2$ m //

Ex:

A 1.20 kg piece of cheese is placed on

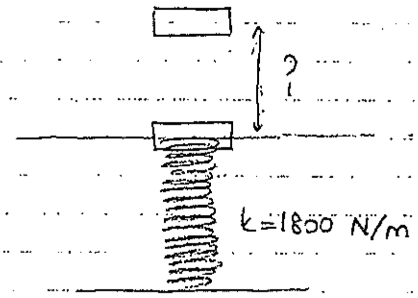
vertical spring of negligible mass and force

stant k=1800 N/m that is compressed

spring 15.0 cm. When the spring is released,

w high does the cheese rise from the initial

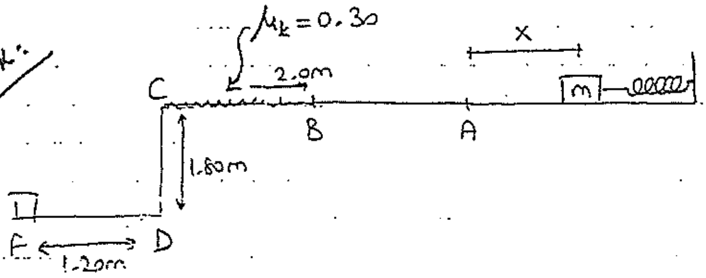
sition?



$U_{1,elastic} = U_{2,gravitational}$

$y_2 = 1.72$ m //

Ex:



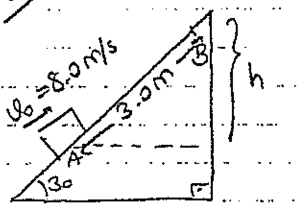
A block of 0.5 kg mass is pushed against a horizontal spring of force constant 200 N/m, compressing it a distance of x. When the spring is released, the block travels first on the horizontal smooth (frictionless) surface to B, passes a rough region of 2.0 m, from B to C, $\mu_k = 0.30$, and finally falls a height of 1.80 m to the point E, at a distance 1.20 m from D

a) what is the speed of the block at point C just before it falls?

b) what is the speed of the block at point B just before it enters the rough region?

c) what is the compression, x of the spring?

Ex:



m=5.0 kg

Determine

a) The change in the block's kinetic energy

b) The change in its potential energy

c) The frictional force exerted on it

d) what is the coefficient of kinetic friction (μ_k)

a) $\Delta K = K_f - K_i = -\frac{1}{2}m(\vartheta_0)^2 = 160$ J

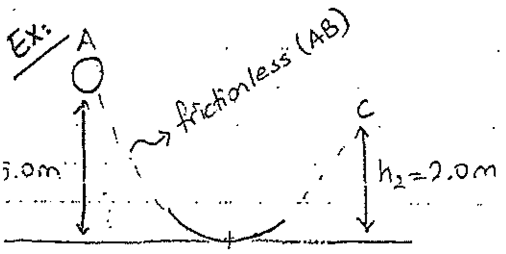
b) $\Delta U = U_{f(B)} - U_{i(A)} = mg\sin\theta$

c) $\Delta K + \Delta U = -f_k \cdot S$ $\quad f_k = \frac{-(\Delta K + \Delta U)}{S}$

d) $f_k = \mu_k N$

$N - mg\cos\theta = 0$ $\Big\}$ $\mu_k = \frac{f_k}{mg\cos\theta}$

Ex:



Starting from rest at point A, mass of ball is.

$m=0.5\ kg$

B_C is rough.

a) Find the speed of the ball at B.

Use conservation of energy.

$E_i=E_f$

$\cancel{K_A}+U_A=K_B+\cancel{U_B}$

$U_A=K_B$

$mgh_1=\frac{1}{2}m(\vartheta_B)^2 \Rightarrow \boxed{(\vartheta_B)^2=\frac{2\cancel{m}gh_1}{\cancel{m}}}$ //

b) If the ball comes to rest at C. Find the energy lost due to friction as it moves on B to C.

$W_{NC}=\Delta K+\Delta U$ $\quad W_{NC}=(K_f-K_i)+(U_f-U_i)$

work done by conservative forces

$W_{NC}=-\frac{1}{2}m(\vartheta_B)^2+mgh_2$

$W_{NC}=-14.7J$ //

POWER

$\boxed{P=\frac{dE}{dt}}$ → joule, → s (Definition of power)

☆ The time rate of energy is called instantaneous power. P is defined as

If an external force applied (W_F)

The average power $P_{avg}=\frac{W}{\Delta t}$

The instantoneous power is the limiting value of the average power as $\Delta t \to 0$

$P=\lim_{\Delta t\to 0}\frac{W}{\Delta t}=\frac{dW}{dt}$

$P=\frac{\vec{F}\cdot d\vec{r}}{dt}$ $\vec{\vartheta}$ $\quad \boxed{P=\vec{F}\cdot\vec{\vartheta}}$

Infinitesimal value of work $dW=\vec{F}\cdot d\vec{r}$

The SI unit $\left(\frac{J}{s}\right) \rightarrow 1$ watt.

$1 w = \left(1 \frac{J}{s}\right) = kg.m^2/s^2$

Unit of power ⇒ 1 horse power

$1 hp = 746 w$

1 kilowatt.hour is the energy transferred 1 h at the constant rate of $1Kw = 10^3 J/S$

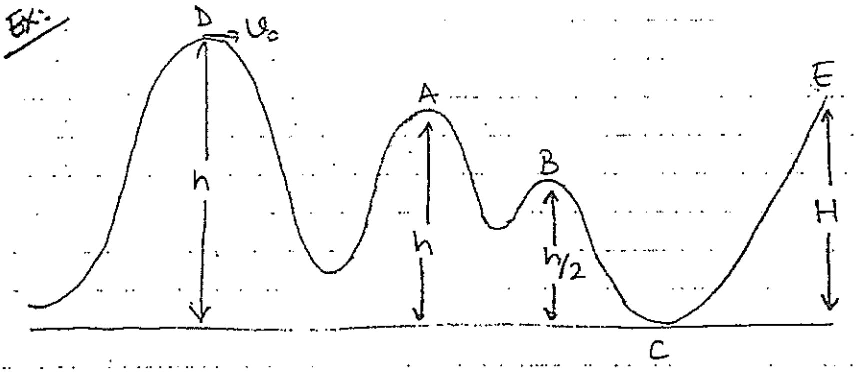
$|dE| = P.dt$

$1 kw.h = (10^3 w).(3600 s)$

$= 3.60 \times 10^5 J$ //

Is a unit of energy

Ex:



Frictionless. Mass is m.

Calculate its speed at points A, B, C

a) $(K)_D + (U)_D = (K)_A + (U)_A$

$v_0 = v_A$ //

b) $(K)_A + (U)_A = (K)_B + (U)_B$

$\frac{1}{2} m v_0^2 + mgh = \frac{1}{2} m v_B^2 + mg\left(\frac{h}{2}\right)$

$v_B = \sqrt{v_0^2 + gh}$ //

c) $(K)_A + (U)_A = (K)_C + (U)_C$

$v_C = \sqrt{v_0^2 + 2gh}$ //

d) How high will it go on the last hill which is too high to cross? H?

#CHAPTER 9#

The Linear Momentum and Collisions:

$\vec{F}$ and $\vec{a}$

$\Sigma \vec{F} = m.\vec{a}$

A new quantity (the linear momentum): $\vec{P}$

$$\vec{P} = \underset{\text{mass}}{m} . \underset{\text{velocity}}{\vec{\vartheta}}$$

It is a vector.

The dimension $M . \frac{L}{T}$

Components of $\vec{P}$:

$P_x = m\vartheta_x$

$P_y = m\vartheta_y$

and

$P_z = m\vartheta_z$

Momentum	Kinetic Energy
$m.\vartheta$	$m.\vartheta^2$
$\Downarrow$	$\Downarrow$
ere is only one form.	transformed

Newton's Second Law:

$$\Sigma \vec{F} = m.\vec{a} = m\frac{d\vec{\vartheta}}{dt} \qquad \left(\vec{a} = \frac{d\vec{\vartheta}}{dt}\right)$$

$$= \frac{d}{dt}(m\vec{\vartheta})$$

$$\boxed{\Sigma \vec{F} = \frac{d\vec{P}}{dt}}$$

→ Newton's Second Law

→ It is more general form of 2nd law.

Conservation of Linear Momentum:

$\vec{P}_1$ and $\vec{P}_2$

$\vec{P} = \vec{P}_1 + \vec{P}_2$ $\qquad \frac{d\vec{P}}{dt} = 0$

$\Sigma \vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(\vec{P}_1 + \vec{P}_2)$ $\qquad \vec{P} \rightarrow$ constant

for an isolated system $\qquad \vec{P}_f = \vec{P}_i$

$\Sigma \vec{F} = 0$

The momentum of the system is conserved, not necessarily the momentum of an individual particle.

Mathematically,

$\vec{P}_{total} = \vec{P}_1 + \vec{P}_2 = \text{constant}$

$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}$ (The isolated system)

In component form, the total momentum in each direction are independently conserved.

$P_{1ix} + P_{2ix} = P_{1fx} + P_{2fx}$

$P_{1iy} + P_{2iy} = P_{1fy} + P_{2fy}$

and

$P_{1iz} + P_{2iz} = P_{1fz} + P_{2fz}$

Forces and Conservation of Momentum:

The forces are not specified as conservative or non-conservative.

There is no indication if the forces are constant or not.

The only requirement is that the forces must be internal to the system.

Impulse and Momentum:

The momentum of a system changes if a net force from environment acts on the system.

From Newton's Second Law

$$\Sigma \vec{F} = \frac{d\vec{P}}{dt}$$

$$d\vec{P} = \Sigma \vec{F}\, dt$$

Integrate

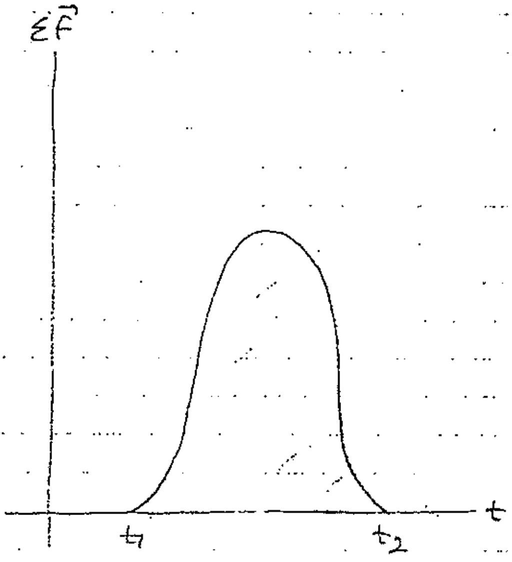
$$\int_{\vec{P}_i}^{\vec{P}_f} d\vec{P} = \int_0^t \Sigma \vec{F} dt$$

$$\underbrace{\vec{P}_f - \vec{P}_i}_{\Delta\vec{P}} = \int_0^t (\Sigma \vec{F}) dt$$

$$\boxed{\Delta\vec{P} = \vec{I}}$$

$$\underbrace{\vec{I} = \int_0^t (\Sigma \vec{F}) dt}$$

The integral is called the impulse.



$$\boxed{\Delta\vec{P} = \vec{I}}$$

This equation expresses the impulse-momentum theorem.

The impulse is a vector quantity.

Impulse is not a property of a particle, rather it is a measure of the degree to which the external force changes the particle's momentum.

$|\vec{I}|$ → The area under the force-time curve.

Ex:

A 50 gr ball is thrown from ground level into the air with an initial speed of 16 m/s an angle of 30° above the horizontal.

a) What are the values of the kinetic energy of the ball initially and just before it hits ground?

$K_i = K_f = \frac{1}{2} m (v_i)^2$

b) Find the corresponding values of the linear momentum.

$P_i = m.v_i$

$\Delta P = ?$

$\Delta P = 2.P_i . \sin\theta$

c) Show that the change in linear momentum is just equal to the weight of the ball multiplied by the time of flight.

$$\Delta P = F.\boxed{t}$$

$$= m.g.\left(\frac{2v_i.\sin\theta}{g}\right)$$

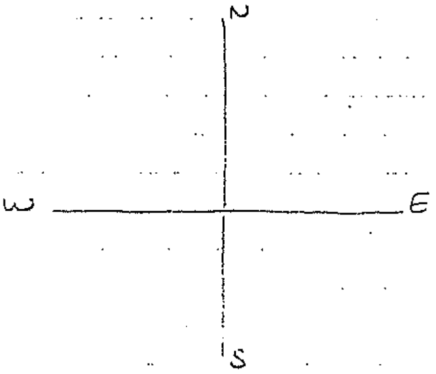
$$t = \frac{v_0}{g}$$

$$2t = \frac{2v_0}{g} \rightarrow t_{flight} = \frac{2v_0}{g}$$

Ex:

A 1400 kg car moving at 5.3 m/s

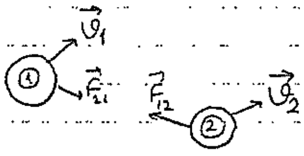
After completing a 90° right turn to positive x-direction in 4.6 s, the inattentive operator drives into a tree, which stops the car in 350 ms.



In unit vector notation, what is the impulse on the car

a) due to the turn and

b) due to the collision?



Newton's Third Law:

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\vec{F}_{21} + \vec{F}_{12} = 0$$

$$m_1\vec{a}_1 + m_2\vec{a}_2 = 0$$

$$\frac{d\vec{v}_1}{dt} \quad \frac{d\vec{v}_2}{dt} \quad m \rightarrow \text{constant}$$

$$m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} = 0$$

$$\frac{d}{dt}(\underbrace{m_1\vec{v}_1}_{\vec{P}_1} + \underbrace{m_2\vec{v}_2}_{\vec{P}_2}) = 0 \quad \frac{d\vec{P}}{dt} = 0 \quad \vec{P} \text{ constant.} \quad \vec{P}_i = \vec{P}_f$$

$$\frac{d}{dt}(\vec{P}_1+\vec{P}_2)=0$$

$$\vec{P}_{total}=\vec{P}_1+\vec{P}_2$$

isolated system ⇒ $\frac{d\vec{P}_{tot}}{dt}=0 \rightarrow \vec{P}$ constant

$$\Sigma\vec{F}=\frac{d\vec{P}}{dt}$$

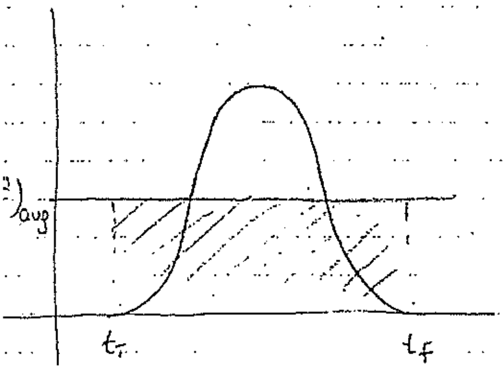
Impulse-Momentum Theorem:

- The time averaged the net force

$$(\Sigma\vec{F})_{avg}=\frac{1}{\Delta t}\int_{t_i}^{t_f}\Sigma\vec{F}.dt$$

$\Delta t=t_f-t_i$

$$\vec{I}=(\Sigma\vec{F})_{avg}.\Delta t$$



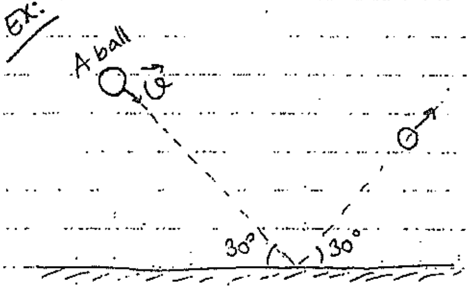
A particle (mass of m)

$\vec{F}$ constant

$$(\Sigma\vec{F})_{avg}=\Sigma\vec{F}$$

$$\vec{I}=\Sigma\vec{F}.\Delta t$$

Ex:



$\vec{v}=6.0$ m/s

The ball rebounds with the same speed and angle

t → 10 m/s → it is in contact with the ball

a) What is the impulse on the ball from the wall?

$$\Delta\vec{P} = \vec{I}$$

$$\vec{I} = m\vec{v}_f - m\vec{v}_i$$

$$\vec{v}_i = v\cos\theta_i - v\sin\theta_i \quad = 5.2\hat{i} - 3.0\hat{j}$$

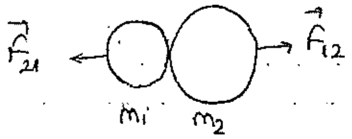
$$\vec{v}_f = v\cos\theta_f + v\sin\theta_f \quad = 5.2\hat{i} + 3.0\hat{j}$$

$$\vec{I} = m\vec{v}_f - m\vec{v}_i$$

b) What is the average force on the wall from the ball?

$\vec{F}_{avg} = 180\hat{j}$ N

Collisions in one Dimension:



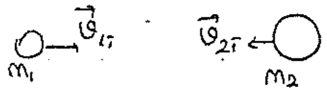
Momentum is conserved.

Types of Collisions:

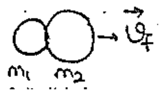
Momentum and kinetic energy are conserved if the collision is elastic collision.

Momentum is conserved, but kinetic energy is not conserved if the collision is inelastic collision

Perfectly Inelastic Collision:



Before collision



After collision

Conservation of momentum:

$$\vec{P}_i = \vec{P}_f$$

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1+m_2)\vec{v}_f$$

$m_2 \rightarrow$ is at rest

$$K_i = \frac{1}{2}m_1 v_{1i}^2$$

$$K_f = \frac{1}{2}(m_1+m_2)v_f^2$$

$$K_f = \frac{1}{2}(m_1+m_2)\left(\frac{m_1 v_{1i}}{m_1+m_2}\right)^2$$

$$K_f = \frac{\frac{1}{2}\cdot(m_1\cdot v_{1i})^2}{m_1+m_2}$$

$$\boxed{\frac{K_f}{K_i} = \frac{m_1}{m_1+m_2}}$$

$\Downarrow$

Perfectly inelastic collision

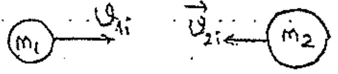
m_2 is initially at rest $\Rightarrow$ momentum is conserved.

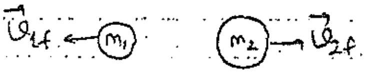
Elastic collision:	Inelastic collision:
K_E is conserved.	Some K_E lost

Perfectly inelastic collisions $\rightarrow$ Bodies have the same velocity after collision.

Elastic Collisions:

Consider two particle system.





Before collision move separately

Conservation of momentum:

Conservation of E_k

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\frac{1}{2}m_1(v_{1i})^2 + \frac{1}{2}m_2(v_{2i})^2 = \cancel{\frac{1}{2}}m_1(v_{1f})^2 + \cancel{\frac{1}{2}}m_2(v_{2f})^2$$

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \qquad (*)$$

$$m_1\cdot(v_{1i} - v_{1f}) = m_2\cdot(v_{2f} - v_{2i}) \qquad (**)$$

Divide Eq. (*) by Eq (**)

$$\boxed{v_{1i} + v_{1f} = v_{2i} + v_{2f}}$$

or

$$\boxed{v_{1i} - v_{2i} = -(v_{1f} - v_{2f})}$$

$\downarrow$

Relative velocities

$v_{1i} = v_{2f} - v_{1f}$

In a straight line elastic collision of two bodies, the relative velocities before and after collision have the same magnitude but opposite direction.

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i}$$

→ If $m_1 = m_2$

$\boxed{v_{1f} = v_{2i}}$ and $\boxed{v_{2f} = v_{1i}}$

→ If $v_{2i} = 0$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i}$$

$$v_{2f} = \left(\frac{2m}{m_1 + m_2}\right) v_{1i}$$

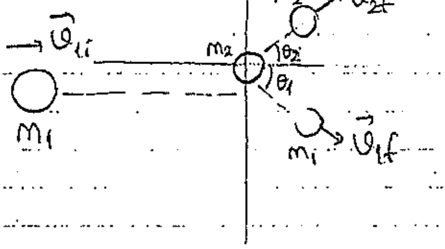
→ If $m_1 >> m_2$ and $v_{2i} = 0$

$v_{1f} = v_{1i}$ and $v_{2f} = 2v_{1i}$

→ If $m_2 >> m_1$

$v_{1f} = -v_{1i}$ $v_{2f} = 0$

Collisions in Two Dimensions:



Before collision for m_1 momentum: $m_1 . v_{1i}$

$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}$

$\vec{P} = m\vec{v}$

$K_{1i} + K_{2i} = K_{1f} + K_{2f}$

If the target is at rest

m_2

$\vec{v}_{2i} = 0$

In x direction:

Conservation of linear momentum:

$m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$ ①

In y direction:

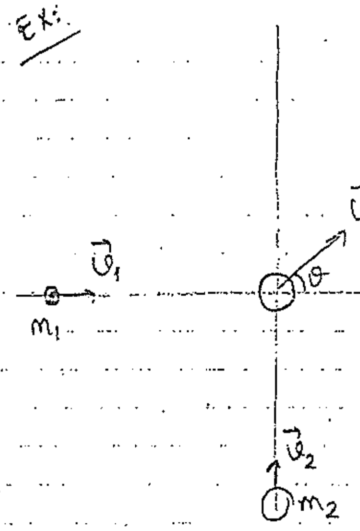
Conservation of linear momentum:

$0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2$ ②

Kinetic energy:

$\frac{1}{2} m_1 (v_{1i})^2 = \frac{1}{2} m_1 (v_{1f})^2 + \frac{1}{2} m_2 (v_{2f})^2$ ③

Ex:



$m_1 = 80$ kg

$v_1 = 6$ km/h

$m_2 = 50$ kg

$v_2 = 8$ km/h

Find velocity of couple immediately after the collision

$m_1 . v_1 = (m_1 + m_2) . \cos\theta . v$

$m_2 v_2 = (m_1 + m_2) \sin\theta . v$

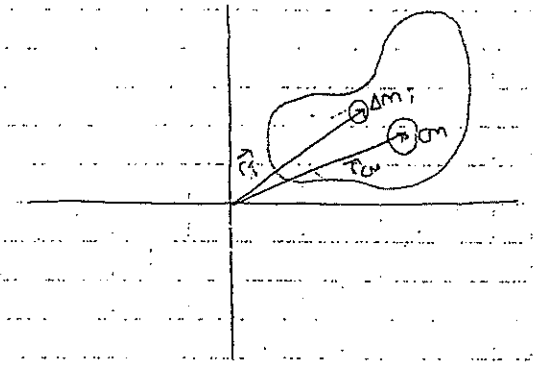
$\tan\theta = \frac{m_2}{m_1} . \frac{v_2}{v_1}$

$\Downarrow$

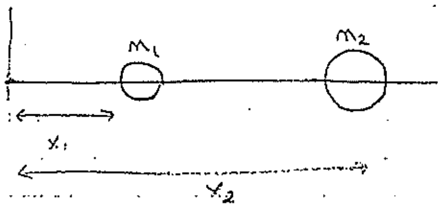
$\theta = 40°$

Center of Mass:

$m_1, m_2, - - -$



If we have two particle system



m_1, m_2

$x_{cm} = ?$ $\quad x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

and $z_{cm} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}$

$$\vec{r} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$$

$M = m_1 + m_2$
↓
total mass

$$x_{cm} = \frac{\sum_i m_i x_i}{\sum_i m_i} \qquad \boxed{M = \sum_i m_i}$$

$$y_{cm} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

$$z_{cm} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

The center of mass in three dimensions:

For a system of particles:

$$\boxed{\vec{r}_{cm} = \frac{1}{M} \sum m_i \vec{r}_i}$$

$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$
↓
The position vector of the i^{th} particle

$$x_{cm} = \frac{\sum_i' \Delta m_i x_i}{\sum \Delta m_i}$$

$$y_{cm} = \frac{\sum_i \Delta m_y y_i}{\sum \Delta m_i}$$

$$z_{cm} = \frac{\sum_i \Delta m_z z_i}{\sum \Delta m_i}$$

$$\sum \Delta m_i = m_1 + m_2 + \cdots + m_n = M$$

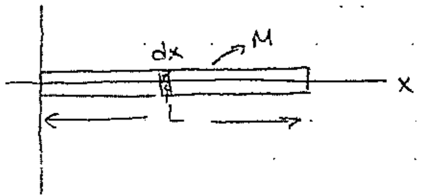
$$x_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum x_i \Delta m_i}{M} = \frac{1}{M}\int x\,dm$$

$$y_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum y_i \Delta m_i}{M} = \frac{1}{M}\int y\,dm$$

$$z_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum z_i \Delta m_i}{M} = \frac{1}{M}\int z\,dm$$

$$\boxed{{}_{cm} = \frac{1}{M}\int r\,dm}$$

★The center of mass of a rod:



Mass is uniformly distributed.

Show that the center of mass of a rod of mass M and length L lies between its ends.

$$\boxed{\lambda = \frac{M}{L}}$$ → The linear mass density

$$x_{cm} = \frac{1}{M}\int x\,(dm)$$

$$\boxed{dm = \lambda . dx}$$

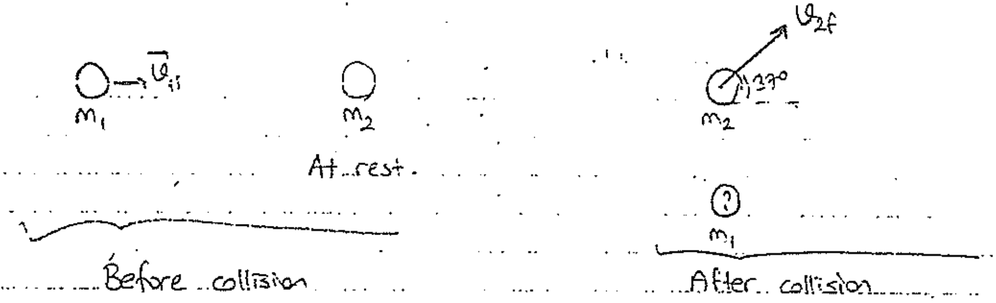
$$x_{cm} = \frac{1}{M}\int_0^L x . \lambda . dx \longrightarrow x_{cm} = \frac{\lambda}{M} . \int_0^L x . dx$$

$$x_{cm} = \frac{\lambda}{M} . \left(\frac{x^2}{2}\right)\Bigg|_0^L$$

$$x_{cm} = \frac{\lambda}{m} . \frac{L^2}{2} = \frac{\frac{M}{L}}{m} . \frac{L^2}{2}$$

$$\boxed{x_{cm} = \frac{L}{2}}$$

Ex: Two pucks colliding. Frictionless surface.



$m_1 = 0.2$ kg

$m_2 = 0.3$ kg

$v_{1i} = 2.0$ m/s

$v_{2i} = 0$ $\quad v_{2f} = 1.0$ m/s

a) Find the final velocity of the puck with mass m_1 in unit vector notation.

$\vec{v}_{1i} = 2.0\hat{i}$ (m/s)

$\vec{v}_{2f} = v_{2f} \cdot \cos 37 \cdot \hat{i} + v_{2f} \sin 37 \cdot \hat{j}$

$\vec{v}_{2f} = 0.8\hat{i} + 0.6\hat{j}$ (m/s)

Use conservation of momentum:

$m_1 v_{1i} = m_1 \vec{v}_{1f} + m_2 v_{2f} \cos 37° \longrightarrow (0.2)(2.0\hat{i}) = 0.2\vec{v}_{1f} + 0.3(0.8\hat{i} + 0.6\hat{j})$

$0 = m_1 v_{1f} + m_2 v_{2f} \sin 37°$ $\quad \Downarrow \vec{v}_{1f} = (0.8\hat{i} - 0.9\hat{j})$ (m/s)

b) Is the collision elastic or inelastic?

$\Delta K = K_f - K_i$

$K_i = \frac{1}{2} m_1 (v_{1i})^2$

$K_f = \frac{1}{2} m_1 (v_{1f})^2 + \frac{1}{2} m_2 (v_{2f})^2$

c) Find the center of mass velocities

v_{cmi} before the collision, and v_{cmf} after collision

$\vec{v}_{cmi} = \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{m_1+m_2}$ $\qquad \vec{v}_{cmf} = \frac{m_1\vec{v}_{1f} + m_2\vec{v}_{2f}}{m_1+m_2}$ $\quad v_{1f} = (0,8\hat{i} - 0,9\hat{j})$ m/s ; $v_{2f} = (0,8\hat{i} + 0,6\hat{j})$ m/s

$= \frac{(0.2)\cdot(2.0\hat{i})}{(0.2+0.3)} = 0.8\hat{i}$ (m/s) // $\qquad \vec{v}_{cmf} = 0.8\hat{i}$ (m/s) //

d) Find the impulse on the puck with mass m_1 in unit vector notation.

$\Delta\vec{P} = \vec{I}$ Impulse – momentum theorem.

$\vec{I}_1 = m_1\vec{v}_{1f} - m_1\vec{v}_{1i}$ $\quad \vec{I}_1 = (-0.24\hat{i} - \underline{0.18}\hat{j})$ (kg.m/s)

Velocity and Momentum of a System of Particles:

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M} \cdot \sum_{i=1}^{n} m_i \cdot \vec{v}_i$$

$$\vec{r}_{cm} = \frac{1}{M} \sum m_i \vec{v}_i$$

$$M\vec{v}_{cm} = \boxed{\sum_{i=1}^{n} m_i v_i} \rightarrow \vec{P}_{total}$$

$$M\vec{v}_{cm} = \boxed{\sum_{i=1}^{n} \vec{P}_i} \nearrow$$

☆ The total linear momentum of the system equals the total mass multiplied by

e velocity of the center of mass of the system.

$\vec{a}_{cm}$

$$\Sigma\vec{F}_{ext} = m \cdot \vec{a}_{cm}$$

The impulse imparted to the system by external forces

$$\int \Sigma F_{ext}\, dt = M \int dv_{cm}$$

$$\boxed{\Delta\vec{P} = \vec{I}}$$

$$M\vec{v}_{cm} = \vec{P}_{total} = \text{constant}$$

$$\boxed{\Sigma F_{ext} = 0} \leftarrow$$

For isolated system of particles.

Acceleration and Force in a System of Particles:

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt}$$

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\sum_{i} \underbrace{m_i \vec{a}_i}_{\vec{F}_i}$$

$$M\vec{a}_{cm} = \sum_{i}^{n} m_i a_i$$

$$M\vec{a}_{cm} = \sum_{i=1}^{n} \vec{F}_i$$

Finally, if the net external force is zero on a system;

$$\sum \vec{F}_{ext} = 0$$

$$M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = 0$$

$$M\vec{v}_{cm} = \vec{P}_{tot} \Rightarrow \text{constant}$$

EX: A 2.0 kg particle has velocity $\vec{v}_1 = 2.0\hat{i} - 10t\hat{j}$, where t is in second. A 3.0 kg particle moves with constant velocity of $\vec{v}_2 = 4.0\hat{i}$ m/s at $t = 0.5$ s. Find

a) velocity of the center of mass.

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

$$\vec{v}_{cm}(t = 0.5\ s) = 3.2\hat{i} - 2\hat{j}\ (m/s)$$

$$\vec{v}_{cm} = \frac{2.0(2.0\hat{i} - 10t\hat{j}) + 3.0(4.0\hat{i})}{(2.0 + 3.0)}$$

b) Find the acceleration of the cm.

$$\vec{a}_{cm} = \frac{d\vec{v}}{dt}$$

$$\vec{a}_{cm} = -4.0\hat{j}\ (m/s^2)$$

c) Find the total momentum of the system.

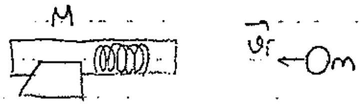
$\vec{P} = \vec{P}_1 + \vec{P}_2$

$\vec{P}_1 = m_1 \vec{v}_1$

$\vec{P}_2 = m_2 \vec{v}_2$

$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2$

f)



The barrel of a spring gun of mass M initially at rest.

The ball sticks in the barrel at the point of maximum compression of the spring. No chanical energy is dissipated by friction.

What is the speed of the spring gun after the ball comes the rest in the barrel?

v → the speed of ball-gun system

$m v_i = (m+M) v$

$v = \frac{m}{m+M} v_i$ //

What fraction of the initial kinetic energy of the ball is stored in the spring?

$K_i = \frac{1}{2} m_1 v_{1i}^2$

Final kinetic energy

$K_f = \frac{1}{2} (m+M) v^2$

$K_f - K_i \longrightarrow U_s = K_i - K_f = \frac{1}{2} m_1 v_{1i}^2 - \frac{1}{2} (m+M) \left(\frac{m v_{1i}}{m+M}\right)^2$

U_s → the spring

$\frac{U_s}{K_i} = \frac{M}{m+M}$ //

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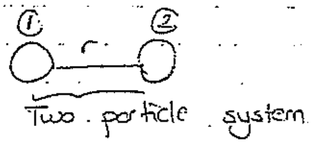
$U(r) = \frac{A}{r}$ $\quad \vec{F_r} = ?$



$$F_r = -\frac{\partial U(r)}{\partial r} = -\frac{d}{dr}\left(\frac{A}{r}\right)$$

$\boxed{F_r = \frac{A}{r^2}} \rightarrow$ The force of repulsion

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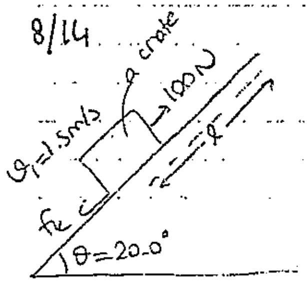

$\mu_k = 0.4$ $\quad l = 5.0$ m

a) How much work $\leftarrow \vec{F_g}$

$W_g = ?$

$W_g = \vec{F_g} \cdot \vec{l}$

$W_g = m.g.l.\cos(90 + \underset{20.0°}{\theta})$

$= -m.g.l.\sin(20.0°)$

b) $\boxed{\Delta E_{int}} \rightarrow$ internal energy

$\Delta E_{int} = f_k . l$ $\quad \Sigma F_y = m a_y \quad a_y = 0$

$\Sigma F_y = 0$

$f_k = \mu_k . N$ $\quad N - mg\cos\theta = 0$

$= \mu_k . (m.g.\cos\theta)$ $\quad N = mg\cos\theta$

$\Delta E_{int} = (\mu_k . m . g . \cos\theta) . l$

c) $F_A = 100$ N

$W_{F_A} = \vec{F_A} . \vec{l}$

$= F_A . l = 500$ J //

d) What is the change in kinetic energy of the crate?

$\Delta K = \Sigma W_{\substack{other \\ forces}} - \Delta E_{int}$

$\boxed{\Delta K = W_{F_A} + W_g - \Delta E_{int}}$

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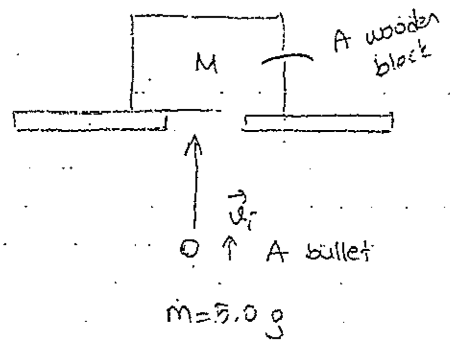


The block and bullet rise to a max. height of 22.0 cm

Bullet_Block_Earth system

$\vec{P}_{1i}+\vec{P}_{2i}=\vec{P}_{1f}+\vec{P}_{2f}$

$m v_{1i}+0=(m+M)v$

$$\boxed{v_{1i}=\frac{(M+M)}{m}\cdot v}$$

$\Delta K+\Delta U=0$

$K_i+\underset{0}{\cancel{U_i}}=\cancel{K_f}+U_f$

$\frac{1}{2}(m+M)v^2=mgh$

$v=\sqrt{2gh}$

3.

A system consists of A and B.

$m_A=2.0$ kg

$m_B=4.0$ kg

$A \rightarrow \vec{v}_A$ constant

$B \rightarrow \vec{v}_B(t)=(2t\hat{i}+3\hat{j})$ m/s

At time t=0 the center of mass of this two particle system is at the origin (x=0, y=0, z=0)

c.m. reaches the point P(x=25 m, y=12 m, z=0) in 6 s. Find $\vec{a}_{cm}$

$\Sigma\vec{F}_{ext}=M\cdot\vec{a}_{cm}$

$M=m_A+m_B$

$M\cdot\vec{a}_{cm}=m_A\cancel{\vec{a}_A}+m_B\cdot\vec{a}_B \longrightarrow a_B=\frac{dv_B}{dt}=2\hat{i}$

$$\boxed{\vec{a}_{cm}=\frac{m_B\cdot\vec{a}_B}{m_A+m_B}}$$

$\vec{a}_{cm}=\frac{4\cdot 2\hat{i}}{2+4}=\frac{4}{3}\hat{i}$ m/s²

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

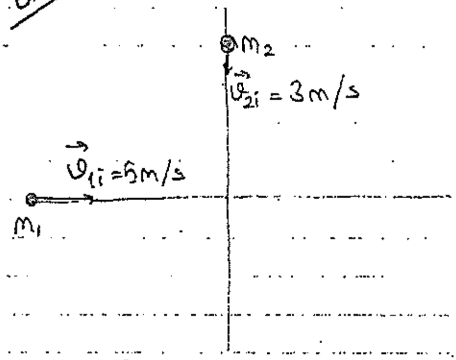
or

$M = m_1 + m_2$

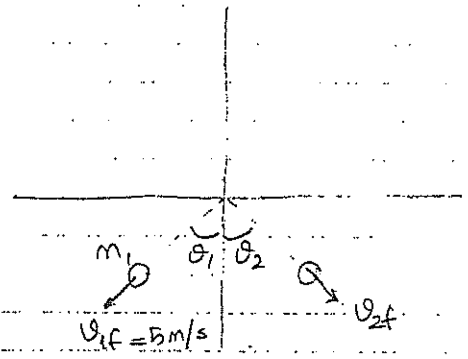
$$M v_{cm} = \overbrace{m_1 v_1}^{P_1} + \overbrace{m_2 v_2}^{P_2} \qquad P_{tot} = P_1 + P_2$$

$$\vec{P} = M \cdot \vec{v}_{cm}$$

Ex:



Before collision



After collision

$m_1, \vec{v}_{1i}, m_2, \vec{v}_{2i}$ $\qquad$ $\vec{v}_{1f}, \vec{v}_{2f}, \theta_1, \theta_2$

→ Conservation of momentum:

in x-direction

$m_1 v_i = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2$

in y-direction:

$m_2 v_2 = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$

→ K_E lost
↓
K_E lost $= K_i - K_f$

→ $\vec{P}_{total} = (m_1 + m_2)\,\vec{v}_{cm}$

Conservation of momentum:

Conservation of K_E

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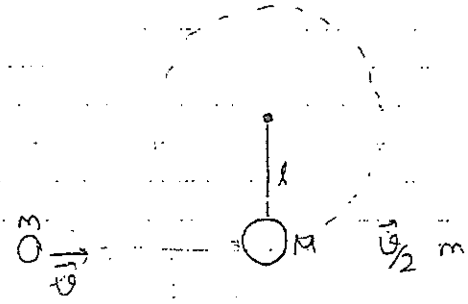

$m v = m\frac{v}{2} + M.v_B$

At the block $\frac{1}{2}.M.v_B^2 = M.g.(2l)$

$v_B = 2\sqrt{gl}$

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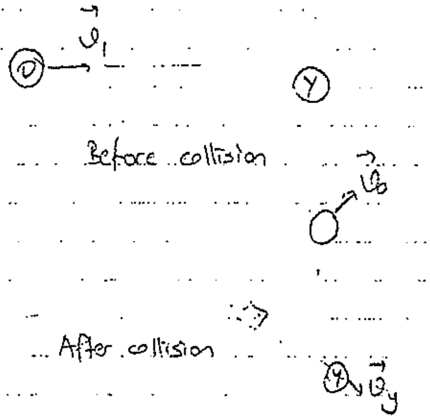


$m_1 = m_2$ Before collision

After collision

$P_{xf} = P_{xi}$

$P_{yf} = P_{yi}$

$\cancel{m}v_1 = \cancel{m}v_0\cos\theta + \cancel{m}v_y\cos(90° - \theta)$

$v_1 = v_0\cos\theta + v_y\cos(90° - \theta)$

$0 = \cancel{m}v_0\sin\theta - \cancel{m}v_y\sin(90° - \theta)$

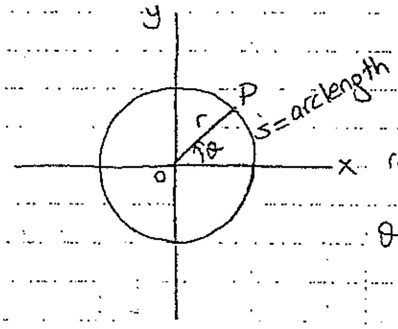
$v_0\sin\theta = v_y\underbrace{\sin(90° - \theta)}_{\cos\theta}$

#CHAPTER 10

Rotation of a Rigid object About a Fixed Axis:

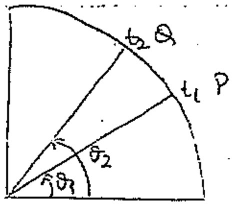
Motion
↙ ↘
Translational Rotational
\+
Rolling motion

Rigid body → definite shape and size



reference line OP

θ = angular position



$\Delta\theta = \theta_2 - \theta_1$. Angular displacement

$$\boxed{\theta = \frac{s}{r}}$$

1 complete revolution (360°)

1 rad $\frac{360°}{2\pi} = 57.3°$

2π rad $= 360°$
π rad $= 180°$
$\frac{\pi}{2}$ rad $= 90°$

Average angular velocity: (ω)

$$\boxed{\omega_{avg} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{t_2 - t_1}}$$

Instantaneous angular speed:

$$\boxed{\lim_{\Delta t \to 0} \omega = \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}}$$

$$\boxed{\begin{array}{c} x \to \theta \\ v \to \omega \\ a \to \alpha \end{array}}$$

Unit of ω: rad/s

1 rev = 2π rad

1 rev/s = 2π rad/s

1 rev/min = $\frac{2\pi r}{60}$ rad/s = 1 rpm

★ If instantaneous angular speed changes

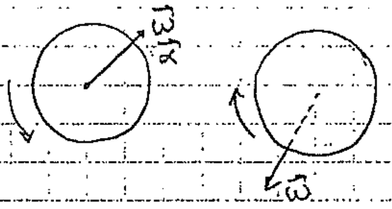
$$\boxed{\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1}}$$ (angular acceleration) rad/s^2

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} //$$

$$\alpha = \frac{d\omega}{dt}$$

$$\int_{v_0}^{v} d\omega = \int_0^t \alpha\, dt \qquad \omega - \omega_0 = \alpha . t$$

$$★\boxed{\omega = \omega_0 + \alpha . t}$$



$$\omega = \frac{d\theta}{dt}$$

$$\int_{\theta_0}^{\theta} d\theta = \int_{0}^{t} \underset{(\omega_0 + at)}{\omega} dt$$

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2} a t^2$$

$$\boxed{\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2}$$ ☆

$$t = \frac{\omega - \omega_0}{a}$$

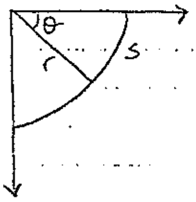
$$\theta = \theta_0 + \omega_0 \left(\frac{\omega - \omega_0}{\alpha}\right) + \frac{1}{2}\alpha\left(\frac{\omega - \omega_0}{\alpha}\right)^2$$

$$\boxed{\omega^2 - \omega_0^2 = 2\alpha\,\Delta\theta}$$

$$\boxed{\theta - \theta_0 = \left(\frac{\omega + \omega_0}{2}\right) t}$$

Relation between Angular and Linear Quantities:

$$s = r\theta$$



$$\frac{ds}{dt} = r . \frac{d\theta}{dt}$$

$$\boxed{\vartheta = r . \omega}$$

a_t = The tangential acceleration

$$a_t = \frac{d(r\omega)}{dt} = r . \frac{d\omega}{dt}$$

$$\boxed{a_t = r . \alpha}$$

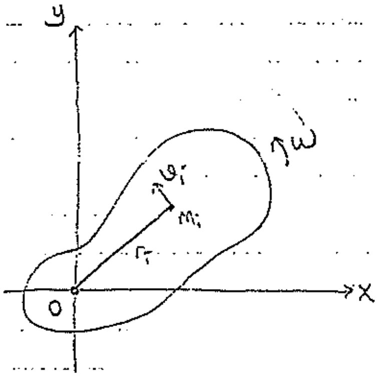
$$a_r = \frac{\vartheta^2}{r} = \frac{\omega^2 r^2}{r}$$

$$\boxed{a_r = \omega^2 . r}$$

$$a = \sqrt{a_t^2 + a_r^2}$$

$$a = \sqrt{r^2(\alpha^2 + \omega^4)}$$

$$\boxed{a = r\sqrt{\alpha^2 + \omega^2}}$$



$$K_i = \frac{1}{2} m_i v_i^2$$

$$v_i = r_i \cdot \omega \nearrow$$

$$K = \sum_{i=1}^{n} K_i = \sum_{i=1}^{n} \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} \cdot \underbrace{\left(\sum_i m_i r_i^2 \right)}_{I} \omega^2$$

☆ $\boxed{K = \frac{1}{2} I \omega^2}$

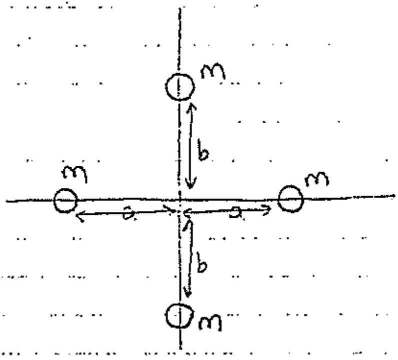
the moment of inertia (depends on the mass)

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots$$

☆ $\boxed{I = \sum_i m_i r_i^2}$

Ex:

Four tiny sphere are fastened to the ends of two rods of negligible mass lying in x-y plane.



If the system rotates about the y axis with an angular speed ω.

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

$$I = Ma^2 + Ma^2 = 2Ma^2$$

$$K = \frac{1}{2} I \omega^2 = \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot M \cdot a^2 \cdot \omega^2 = M \cdot a^2 \cdot \omega^2$$

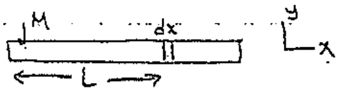
☆ Let's consider an object having mass is distributed in 3-space.

$$I \sim \sum_{i=1}^{n} r_i^2 \Delta m_i \longrightarrow I = \lim_{\Delta m_i} \sum r_i^2 \Delta m_i = \int_0^m r^2 \, \underline{\underline{dm}}$$

differential mass element

$n \to \infty \quad \Delta m_i \to 0$

☆ A uniform rigid rod



$$\lambda = \frac{M}{L} \qquad dm = \lambda \cdot dx$$

$$I = \int_{-L/2}^{L/2} x^2 (\lambda dx) = \lambda \cdot \frac{x^3}{3} \Big|_{-L/2}^{L/2} = \frac{\lambda}{3} \cdot \left(\left(\frac{L}{2}\right)^3 - \left(\frac{-L}{2}\right)^3 \right) = \frac{\lambda \cdot L^3}{12}$$

$$\boxed{I = \frac{1}{12} M L^2}$$

ω counterclockwise θ positive → ω

clockwise θ negative → −ω

$\lambda = \frac{M}{L}$ $\quad \sigma = \frac{M}{\text{area}}$ $\quad \rho = \frac{M}{\text{Vol}}$

Ex:

A wheel rotates with angular acceleration α, $\alpha = 4\underset{\text{constant}}{a}\overset{\text{time}}{t^3} - 3\underset{\text{constant}}{b}t^2$

If the wheel has an initial angular speed ω_0, write equations for

a) Angular speed and

b) The angular displacement of the wheel as a function of time.

$$\alpha = \frac{d\omega}{dt} \qquad \int_{\omega_0}^{\omega} d\omega = \int_{0}^{t} a\,dt$$

$$\omega - \omega_0 = at^4 - bt^2$$

$$\omega = \omega_0 + at^4 - \underline{\underline{bt^2}}$$

$$\omega = \frac{d\theta}{dt} \qquad \theta = \theta_0 + \omega_0 t + \frac{at^5}{5} - \frac{bt^4}{4} \;//$$

Ex:

A computer disc drive is turned on starting from rest and has constant angular acceleration.

It took <u>0.750 s</u> for the drive to make its second complete revolution,

a) How long did it take to make the first complete revolution and

b) What would be its angular acceleration in rad/s^2?

a) $\theta = \underset{0}{\cancel{\theta_0}} + \underset{0}{\cancel{\omega_0 t}} + \frac{1}{2}\alpha t^2$

$$\theta = \frac{1}{2}\alpha t^2$$

$$2\pi \text{ rad} = \frac{1}{2}\alpha t^2$$

$$2.2\pi \text{ rad} = \frac{1}{2}\alpha\left(t + \frac{3}{4}\right)^2$$

$$\frac{1}{2} = \left(\frac{t}{t + \frac{3}{4}}\right)^2$$

$$\frac{1}{\sqrt{2}} = \frac{t}{t + \frac{3}{4}}$$

$$t + \frac{3}{4} = \sqrt{2}t \qquad t^2 + \frac{9}{16} + \frac{3t}{2} = 2t^2$$

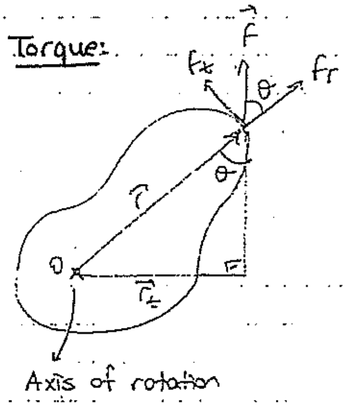
$$t^2 - \frac{3t^2}{2} - \frac{9}{16} = 0$$

$$t = 1.81\text{s} \;//$$

b) $2\pi \text{rad} = \frac{1}{2}\alpha t^2$

$\alpha = ?$ $\longrightarrow 3.84 \text{ rad/s}^2 \;//$

Torque:



$f_r = f\cos\theta$

$f_t = f\sin\theta$

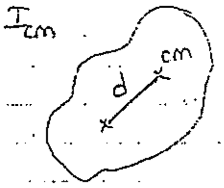
$$\boxed{\Gamma = r.f\sin\theta = r.f_t = r_t.f}$$

↓ the magnitude of torque

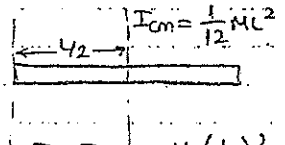
Unit of torque: (N.m)

In vector form: $\boxed{\vec{\Gamma} = \vec{r} \times \vec{F}}$ (cross product)

Parallel Axis Theorem:



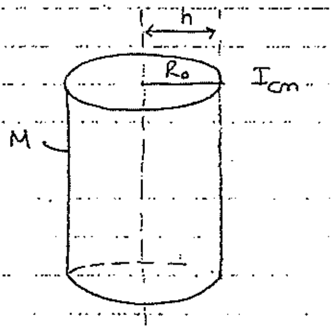
$\boxed{I = I_{cm} + Md^2}$ ★



$$I = I_{cm} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2$$

$$= \frac{1}{3}ML^2 \,//$$



$I = I_{cm} + M\underset{\searrow R_o}{h^2}$

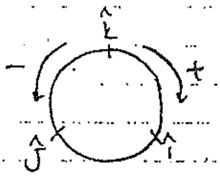
$\vec{C} = \vec{A} \times \vec{B} = AB\sin\theta$

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

if $\vec{A} \parallel \vec{B} \Rightarrow \vec{A} \times \vec{A} = 0$

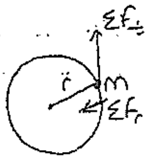
$\Rightarrow \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}$



$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$

$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$



Newton's Second Law:

$\vec{F} = m.\vec{a}$

$\Sigma \vec{F}_t = m\vec{a}_t$ $\quad (v = r.\omega \quad a_T = r.\alpha)$

$= m.r.\alpha$

$\Sigma \Gamma_{net} = r.(m.r.\alpha)$

$= mr^2\alpha$

$$\boxed{\Sigma \Gamma_{net} = I\alpha}$$

$dF_t = r\,dm\,(a_t)$

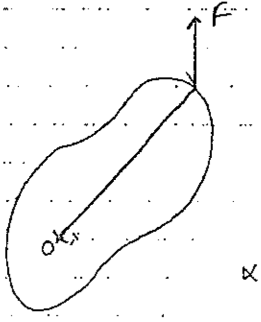
$d\Gamma_{ext} = r.d.F_t$

$= a_t.r.dm$

$$\boxed{d\Gamma_{ext} = \alpha.r^2.dm} \rightarrow \Sigma\Gamma_{ext} = \int d\Gamma_{ext} = \int \alpha r^2 dm$$

$$= \alpha \int r^2 dm$$

$$\Rightarrow \boxed{\Sigma\Gamma_{ext} = I\alpha}$$



$S = r\theta$ $\qquad dw = F.ds$

$ds = rd\theta$ $\qquad dw = F.r.d\theta$

$W = \int dw = \int Frd\theta = \int I\alpha\, d\theta$

$\alpha = \frac{d\omega}{dt}$

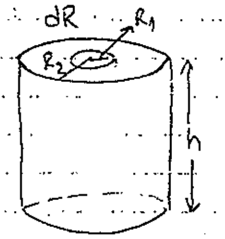
$\alpha\, d\theta = \frac{d\omega}{dt} d\theta$

$W = \int_{w_i}^{w_f} I.w\,dw = \frac{I.w^2}{2}\Big|_{w_i}^{w_f} = W = \frac{1}{2}Iw_f^2 - \frac{1}{2}Iw_i^2$

$$\boxed{W = \Delta K = \frac{1}{2}I\,w_f^2 - \frac{1}{2}Iw_i^2}$$

Ex:

Show that the moment of inertia of a uniform hollow cylinder inner radius R_1, outer radius R_2 and mass M is



$$I = \frac{1}{2}M.(R_1^2 + R_2^2).$$

If the rotation axis is through the center along the axis of symmetry.

$dm = \rho dV \quad \rho = \frac{M}{V} \qquad I = \sum_i^2 m_i r_i^2 \Rightarrow \int r^2 dm$

$dV = (2\pi R)dRh$

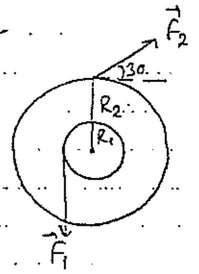
$dm = \rho(2\pi RdRh)$ $\qquad I = \int R^2 dm = \int_{R_1}^{R_2} R^2 (\rho 2\pi R dRh = \underbrace{2\pi\rho h}_{A} \int_{R_1}^{R_2} R^3 dR$

$$I = A.\frac{R^4}{4}\Big|_{R_1}^{R_2} = \frac{A}{4}.(R_2^4 - R_1^4)$$

$V = (\pi R_2^2 - \pi R_1^2)\, h$ $\qquad R_2^4 - R_1^4 = (R_2^2 - R_1^2)(R_2^2 + R_1^2)$

$$\left.\begin{array}{l} I = \frac{\pi \rho h}{2}.(R_2^2 - R_1^2)(R_2^2 + R_1^2) \\ M = \rho \pi.(R_2^2 - R_1^2)\, h \end{array}\right\rangle I = \frac{1}{2} M (R_2^2 + R_1^2)$$

Ex:



Two thin cylindrical wheels of radius $R_1 = 30$ cm and $R_2 = 50$ cm are attached to each other on an axle that passes through the center of each.

Calculate the net torque on the two-wheel system due to the forces shown, $F_1 = F_2 = 50$ N

$$\Gamma_{net} = F_2 \cos 30.R_2 - F_1.R_1$$

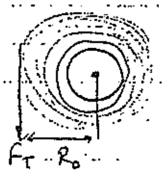
$$= 50.\frac{4}{5}.\frac{1}{2} - 50.\frac{3}{10} = 20 - 15 = 5 \text{ N.m}$$

☆ The power associated with work done by a torque acting on a rotating body:

$$\frac{dw}{dt} = \Gamma \underbrace{\frac{d\theta}{dt}}_{\omega} \qquad \boxed{P = \Gamma.\omega} \quad (\text{in linear } P = F.v)$$

Ex:

A 15.0 N force (represented by $\vec{F_T}$) is applied to a cord wrapped around a pulley of mass $M = 4.0$ kg and radius $R_0 = 33.0$ cm



The pulley is observed to accelerate uniformly from rest to reach an angular speed 30 rad/s in 3 s. If there is a frictional torque $\Gamma_{fr} = 1.10$ mN, determine the moment of inertia of the pulley. (It rotates about its center.)

$\Sigma \Gamma_{net} = F_T.R_0 - \Gamma_{fr}$ $\qquad \Sigma \Gamma = I.\alpha$

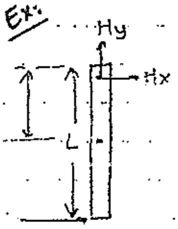
$= 15 \times \frac{33}{100} - 1.10$ $\qquad w_i = 0$

$= 4.95 - 1.10$ $\qquad w_f = 30$ rad/s $\Big\}\ \alpha = \frac{w_f - w_i}{t_f - t_i} = \frac{30 - 0}{3 - 0} = 10$ rad/s^2

$= 3.85$ mN

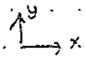
$3.85 = I.10 \qquad I = 0.385$ kgm^2/s

Ex:



A thin rod $M = 0.650$ kg $L = 12.4$ m $\vec{F} = 14.7\hat{i}$

Find the acceleration of its CM and the horizontal force the hinge exerts.



$I = I_{cm} + Md^2 \Rightarrow I = \frac{1}{3}ML^2$ (parallel axis theorem)

$= \frac{1}{12}ML^2$

$\Sigma\Gamma = I\alpha$

$F.L = I.\alpha$

$(14.7)(1.24) = \frac{1}{3}.(0.65).(1,24)^2.\frac{a_{cm}}{(1.24)}.2$

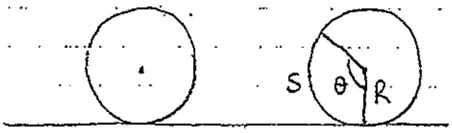
$a_{cm} = \frac{3F}{2m} = 35.0\ m/s^2$

$\alpha R = a_{cm} \quad \alpha\left(\frac{L}{2}\right) = a_{cm} \quad \alpha = \frac{a_{cm}}{\frac{L}{2}}$

$\Sigma F_x = ma_x$

$F_x - H_x = ma_x \quad \boxed{H_x = F_x - ma_x}$

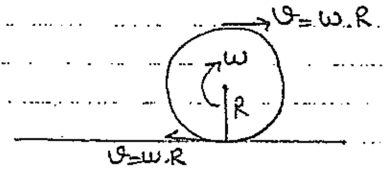
Rolling Motion of a Rigid Body:



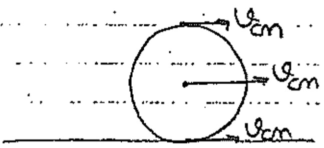
$S = R\theta$

$\frac{dS}{dt} = R.\frac{d\theta}{dt}$

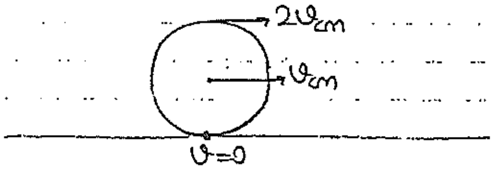
$\boxed{v = R.\omega}$



Pure rotation

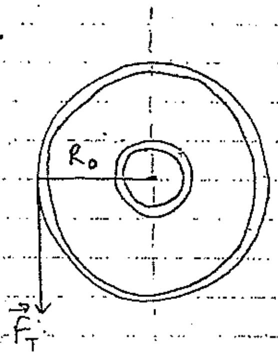


Pure translation



Rolling motion

Ex:



A 15.0 N force (represented by $\vec{F}_T$) is applied to a cord wrapped around a pulley of mass $M = 4.0$ kg and radius $R_0 = 33.0$ cm. The pulley is observed to accelerate uniformly from rest to reach an angular speed of 30 rad/s in 3.0 s. If there is a frictional torque (at the axle) $T_{fr} = 1.10$ m.N, determine the moment of inertia of the pulley. The pulley is assumed to rotate about its center.

$I = ?$

$\Sigma\Gamma = I.\alpha \Rightarrow I = \frac{\Sigma\Gamma}{\alpha} \qquad \alpha = \frac{\Delta\omega}{\Delta t}$

$\Sigma\Gamma = F_T.R_0 - \Gamma_{fr}$

$F_T.R_0 - \Gamma_{fr} = I.\frac{\Delta\omega}{\Delta t}$

$\alpha = \frac{\omega_f - \omega_i}{t_f - t_i}$

$\alpha = \frac{30}{3.0} = 10 \text{ rad/s}^2$

$\underbrace{15.(0.33) - 1.10} = I.10$

$\Sigma\Gamma = 3.85 \text{ m.N.}$

$I = \frac{3.85}{10} \Rightarrow I = 0.385 \text{ kg.m}^2$ //

*Rolling motion of a wheel is a combination of purely translational and purely rotational motions.

$\boxed{I_P = I_{cm} + MR^2}$ → the moment of inertia about a rotation axis through P.

$K_P = \frac{1}{2} I_P \omega^2$

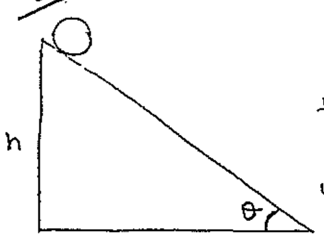
$K_P = \frac{1}{2}(I_{cm} + MR^2)\omega^2$

$K_P = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M \overbrace{R^2\omega^2}^{=\vartheta^2} \qquad (\vartheta_{cm} = \underline{\underline{R.\omega}})$

$\boxed{K_P = \frac{1}{2} I\omega^2 + \frac{1}{2} M\vartheta_{cm}^2}$

*Total kinetic energy of an object undergoing rolling motion is the sum of a rotational K_E about the center of mass and translational K_E of the center of mass.

Ex:



A uniform solid disk and a uniform hoop are placed side by at the top of an incline of height h. If they are released from rest and roll without slipping, determine their speeds when they reach the bottom, which object reach the bottom first?

$I_{disk} = \frac{1}{2}MR^2 \qquad I_{hoop} = MR^2$

Conservation of energy:

$E_i = E_f$

$$\cancel{K_i} + U_i = K_f + U_f$$

$$mgh = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \omega^2$$

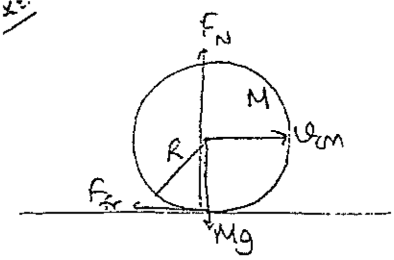
$$mgh = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \left(\frac{v_{cm}^2}{R^2}\right) \Rightarrow v_{cm} = \left[\frac{2gh}{1+\left(\frac{I_{cm}}{mR^2}\right)}\right]^{1/2}$$

$$v_{disk} = \left[\frac{2gh}{1+\left(\frac{\frac{1}{2}MR^2}{MR^2}\right)}\right]^{1/2} = \sqrt{\frac{4gh}{3}}$$

$$v_{hoop} = \left[\frac{2gh}{1+\left(\frac{MR^2}{MR^2}\right)}\right]^{1/2} = \sqrt{gh}$$

$$v_{disk} > \underline{\underline{v_{hoop}}}$$

Ex:



A bowling ball of mass M and radius R is thrown along the level surface so that initially (at t=0) it slides with a linear speed v_0, but it does not rotate. As it slides, it begins to spin, and eventually rolls without slipping.

How long does it take to begin rolling without slipping?

Newton's Second Law:

$$\Sigma F_x = M a_x = -\mu_k \cdot M \cdot g \qquad F_{fr} = \mu_k \cdot M \cdot g$$

$$\underline{\underline{a_x = -\mu_k \cdot g}}$$

$$v = v_0 + a_x \cdot t$$

★ $$\boxed{v_{cm} = v_0 - \mu_k \cdot g \cdot t}$$

$$\omega = \omega_0 + \alpha t$$

Friction force;

1) acts to slow down translational motion of cm.

2) and immediately acts to start the ball rotating clockwise.

$$\underbrace{I_{cm}}_{\frac{2}{5} \cdot M \cdot R^2} \cdot \alpha_{cm} = \Sigma \Gamma = \mu_k \cdot m \cdot g \cdot R$$

$$\alpha_{cm} = \frac{\mu_k . m . g . R}{I_{cm}} \Rightarrow \alpha_{cm} = \frac{5}{2R} \cdot \mu_k \cdot g$$

$(\omega = \omega_0 + \alpha t)$

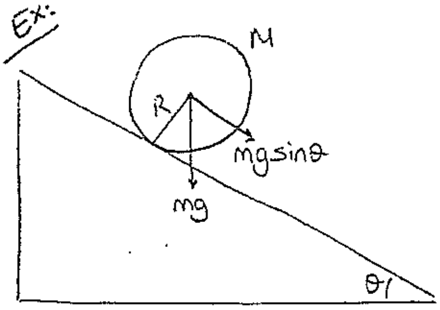
$\downarrow$

$$\omega = 0 + \frac{5}{2R} \cdot \mu_k \cdot g . t$$

$$\boxed{\omega = \frac{5}{2R} \cdot \mu_k \cdot g . t}$$

$$\left.\begin{array}{l} \omega = \frac{5}{2R} \cdot \mu_k \cdot g . t \\ v_{cm} = v_0 - \mu_k . g . t \end{array}\right\rangle \quad v_{cm} = \omega . R \qquad \boxed{t = \frac{2v_0}{7\mu_k g}} //$$

Ex:



Uniform body of mass M and radius R is rolling down a ramp at an angle θ. what is the linear acceleration of the rolling body?

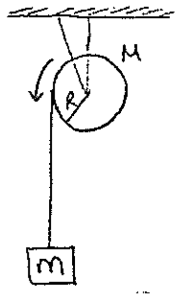
Analyze the motion from Newton's second law.

$\Sigma \vec{F} = m . \vec{a}$

$$\left.\begin{array}{l} \boxed{\Sigma F_x = f_{fr} - Mg\sin\theta = Ma} \\ \Sigma \tau = I_{cm} . \alpha \qquad \alpha = \frac{a}{R} \\ \boxed{-f_{fr} . R = I_{cm} . \frac{a}{R}} \end{array}\right\rangle \left.\begin{array}{l} f_{fr} = \frac{-I . a}{R^2} \\ f_{fr} - Mg\sin\theta = Ma \end{array}\right\rangle \frac{-Ia}{R^2} - Mg\sin\theta = Ma$$

$$a = \frac{-g\sin\theta}{1 + \frac{I_{cm}}{MR^2}} //$$

Ex:



A block of mass m=15 kg is attached to a string which is wrapped several times around a uniform solid cylinder of radius R=10 cm and mass M=60 kg. The cylinder can rotate freely about a horizontal axis through its center. What is the tension T in the string?

$$\Sigma \tau = I_{cm} \cdot \alpha$$

$$\boxed{T.R = I_{cm} \cdot \alpha} \qquad I_{cm} = \frac{1}{2} M R^2 \,/\!/$$

$$\boxed{mg - T = ma}$$

$$\alpha = \frac{a}{R}$$

$$T\cancel{R} = \frac{1}{2} M R^{\cancel{2}} \cdot \frac{a}{\cancel{R}}$$

$$T = \frac{1}{2} Ma \,/\!/$$

$$mg - T = ma$$

$$mg - \frac{1}{2} Ma = ma \Rightarrow a = \frac{mg}{\frac{M}{2} + m} \qquad (g = 10\ m/s^2,\ m = 15\ kg,\ M = 60\ kg)$$

$$a = \frac{10}{3}\ m/s^2 \,/\!/$$

$$T = \frac{1}{2} . M . a = \frac{1}{2} . 60 . \frac{10}{3} \Rightarrow T = 100\ N$$

#CHAPTER 11

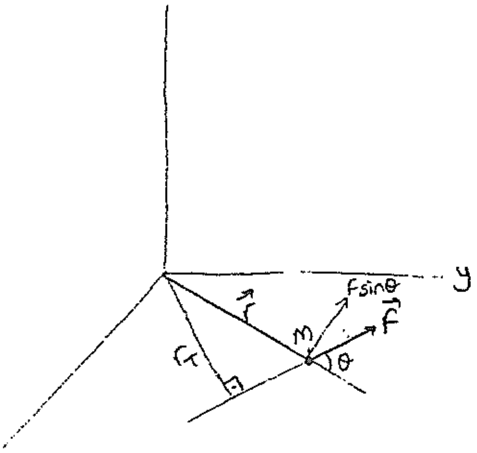
Angular Momentum

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The magnitude of $\vec{\tau}$:

$$\tau = r . F . \sin\theta$$

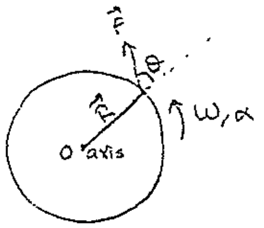
$$\left(\begin{array}{l} \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \\ \vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C}) \\ \frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt} \end{array} \right)$$



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = r_\perp . F$$

$$\tau = r . (F . \sin\theta)$$



The torque due to F will tend to increase

ω and x

$\Gamma = r.F_T$

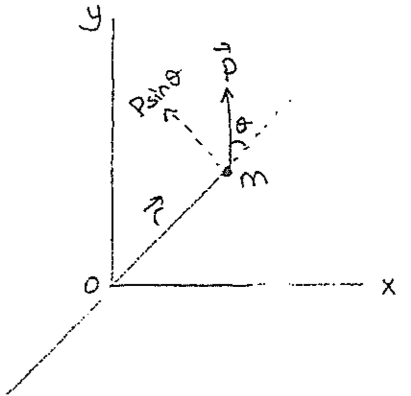
$= r.F.\sin\theta$

Angular Momentum:

$\vec{P} = m.\vec{\vartheta}$ (linear momentum)

Newton's Second Law:

$\Sigma\vec{F} = \frac{d\vec{P}}{dt}$



$\boxed{\vec{L} = \vec{r} \times \vec{P}}$ (Angular momentum)

The magnitude of L; $\boxed{L = r.P.\sin\theta}$

The direction of L is ⊥ to (x,y plane)

F acts on a particle changes its velocity and momentum.

$\vec{L} = \vec{r} \times \vec{P}$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{P}) = \frac{d\vec{r}}{dt}\times\vec{P} + \frac{d\vec{P}}{dt}\times\vec{r}$$

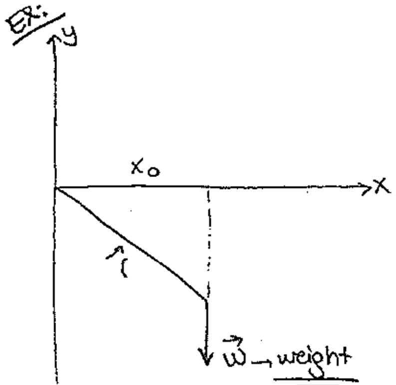
$$= \underbrace{\vec{\vartheta}\times\vec{P}}_{0} + \vec{r}\times\frac{d\vec{P}}{dt} \qquad \vec{\vartheta}\times\vec{P} = \vec{\vartheta}\times(m\vec{\vartheta})$$

$$= m.(\underbrace{\vec{\vartheta}\times\vec{\vartheta}}_{0}) \qquad (\vec{A}\parallel\vec{B} \Rightarrow 0)$$

$$= 0$$

$$\frac{d\vec{L}}{dt} = \vec{r}\times\underbrace{\frac{d\vec{P}}{dt}}_{\vec{F}}$$

$\boxed{\Sigma\vec{\Gamma} = \frac{d\vec{L}}{dt}}$ (for a particle acted on by a net force F)



$\vec{\Gamma} = ?$ acting on the particle at any time with respect to the origin.

$\vec{\Gamma} = \vec{r}\times\vec{\omega}$

$\Gamma = \overbrace{r(mg)\sin\theta}^{x_o}$

$\Gamma = m.g.x_o$ //

The direction into ⊗

$\vec{L} = ?$

$\vec{L} = \vec{r} \times \vec{P}$

$\vec{L} = \vec{r} \times (m.\vec{\vartheta})$

The magnitude of L:

$L = \overbrace{r.(m.\vartheta.\sin\theta)}^{x_0}$

$\underline{\underline{L = m.\vartheta.x_0}} \quad \otimes$

Show that $\vec{\Gamma} = \frac{d\vec{L}}{dt}$

$\Gamma = mg.x_0$

$L = Px_0 = m.\vartheta, x_0$

$\frac{dL}{dt} = m.x_0.\underset{g}{\frac{d\vartheta}{dt}}$

$\frac{dL}{dt} = \underbrace{m.x_0.g}_{\Gamma} \Rightarrow \Gamma = \frac{dL}{dt}$ //

$\vec{L} = \vec{r} \times \vec{P}$

$L = r.P.\sin\theta$

If $\vec{r} \parallel \vec{P}$ $(\theta = 0, 180°) \Rightarrow L = 0$ //

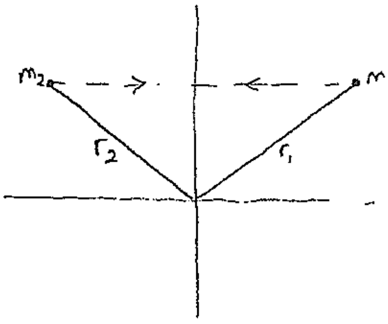
The SI unit of L: $kg.m^2/s$

Angular Momentum of a System of Particles:

$m_1, m_2, ----, m_n$

$L_{tot} = \sum_{i=1}^{n} L_i$

$L_{tot} = L_1 + L_2 + --- + L_n$



Differentiate:

$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{dL_i}{dt}$

$$\boxed{\frac{dL_{tot}}{dt} = \sum \vec{\Gamma_i}}$$

$$\boxed{\frac{dL}{dt} = \sum_{i=1}^{n} \Gamma_i}$$

$$\Sigma\Gamma = \underset{0}{\cancel{\Sigma\Gamma_{int}}} + \Sigma\Gamma_{ext}$$

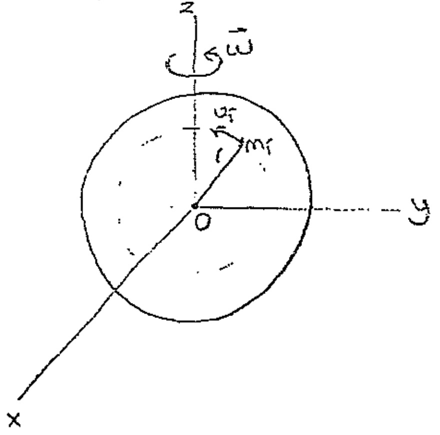
The total angular momentum of a system can vary with time only if a net external torque is acting on the system.

$$\Sigma\vec{\Gamma}_{ext} = \frac{d\vec{L}_{tot}}{dt} \longrightarrow \text{It can be rearranged.}$$

$$\boxed{\int(\Sigma\vec{\Gamma}_{ext})dt = \Delta\vec{L}_{tot}}$$

Angular impulse
Angular momentum theorem

Angular Momentum of a Rotating Rigid Object



$$v_i = r_i . \omega$$

$$L_i = m_i . v_i . r_i$$

$$L_i = m_i . (r_i . \omega) . r_i$$

$$L_i = \underbrace{(m_i . r_i^2)}_{I} . \omega$$

$$L = \sum_i L_i$$

$$L = \sum_i m_i (r_i . \omega) r_i$$

$$L = \left(\sum_i m_i . r_i^2\right) \omega$$

$$\boxed{L = I . \omega}$$

$$L_z = I \omega_z$$

$$L_z = I . \omega_z \qquad \omega_z = \omega$$

Differentiate

$$\frac{dL_z}{dt} = I . \frac{d\omega}{dt}$$

$$\boxed{\frac{dL_z}{dt} = I . \alpha}$$

Angular acceleration relative to the axis of rotation

Analysis Model: Isolated Systems:

$$\boxed{\Sigma \vec{\Gamma}_{ext} = \frac{d\vec{L}}{dt}}$$

$(\Sigma \vec{\Gamma}_{ext} = 0)$

If the system is isolated, then

$\vec{L}_{tot}$ = constant

$$\left(\begin{array}{l} \Sigma \vec{F} = \frac{d\vec{P}}{dt} = 0 \\ \vec{P}_i = \vec{P}_f \end{array} \right)$$

$\vec{L}_i = \vec{L}_f$

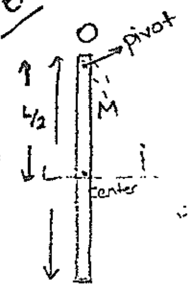
For an isolated system consisting of a number of particles,

$$\vec{L}_{tot} = \underbrace{\Sigma \vec{L}_n}_{constant}$$

$L_i = L_f$

$$\boxed{I_i \omega_i = I_f \omega_f}$$

Ex:



A uniform rod L=1.0 m, M=3.0 kg

Hanging vertically, it can rotate freely around a pivot at its upper point. A bullet of mass m=20 g and speed ϑ=400 m/s hits the rod at its center and becomes embedded in it.

a) Calculate the angular velocity of the rod just after the collision.

Use conservation law: $L_i = L_f$

$L_i = m.\vartheta.\frac{L}{2}$

$L_i = (20\times10^{-3})(400).(0.5)$

$L_i = 4.0$ kg.m²/s

$L_f = I_f.\omega_f$

$I_f = I_{tot}$

Using parallel axis theorem:

$$I_{rod} = \underbrace{I_{cm}}_{I_{cm} = \frac{1}{12}ML^2} + M\left(\frac{L}{2}\right)^2$$

$$I_{rod} = \frac{1}{3}ML^2$$

$$I_{tot} = I_{rod} + I_{bullet}$$

$$\downarrow$$

$$I = \Sigma m_i . r_i^2$$

$$I_{bullet} = m . \left(\frac{L}{2}\right)^2$$

$I_{tot} = 1.0 \ kgm^2$

$L_i = I_{tot} . \omega$

$\omega = \frac{L_i}{I_{tot}} = \frac{11.0}{1.0} = 11.0 \ rad/s$ //

b) Calculate the kinetic energy lost in the collision

$K_i = \frac{1}{2} m \vartheta^2$

$K_f = \frac{1}{2} I_{tot} \omega^2$

$K_i = 1.6 \times 10^3 \ J$

$K_f = 8.0 \ J$

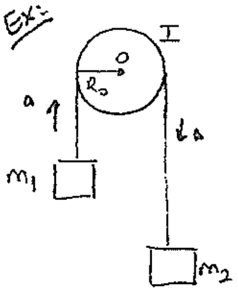
$\Big\rangle \ K_i - K_f = 1592 \ J$ //

c) How high will the center of mass of the rod rise from its initial position after the collision?

$MgH = \frac{1}{2} I \omega^2$

$H = \frac{I\omega^2}{2Mg}$ //

Ex:



An Atwood's machine //

It consists of two masses m_1 and m_2, which are connected by a massless inelastic cord that passes over a pulley. Determine the acceleration of the masses m_1 and m_2 and compare to the situation where the moment of inertia of the pulley is ignored.

$L = I . \omega$

$\vartheta = R . \omega$

$\vec{L} = \vec{r} \times \vec{P}$

$L = m \vartheta r \qquad r = R_0$

$L = m_1 \vartheta R_0 + m_2 \vartheta R_0 + I \cdot \frac{\vartheta}{R_0}$

$L_{tot} = \left(m_1 R_0 + m_2 R_0 + \frac{I}{R_0}\right) \vartheta$

$\vec{\Gamma} = \vec{r} \times \vec{F}$

$\Gamma = r.F$

$\Gamma = m_2 g R_0 - m_1 g R_0$

$\Gamma_{tot} = \frac{dL}{dt}$

$\frac{dL}{dt} = (m_1 + m_2) R_0 \frac{d\vartheta}{dt} + \frac{I}{R_0} \cdot \frac{d\vartheta}{dt} \qquad \boxed{\frac{d\vartheta}{dt} = a}$

$\frac{dL}{dt} = m_2 g R_0 - m_1 g R_0 = (m_2 - m_1) g R_0$

$(m_2 - m_1) g R_0 = \left((m_1 + m_2) R_0 + \frac{I}{R_0}\right) . a$

$$\boxed{a = \frac{(m_2 - m_1) g.}{(m_1 + m_2) + \frac{I}{R_0^2}}}$$

If $I = 0$ the system will move faster. I slows down the system

$\sum \Gamma_{ext} = \frac{d\vec{L}}{dt}$

$\text{If} = 0 \rightarrow \vec{L} = \text{constant}$

$L_i = L_f$

CHAPTER 12

Static Equilibrium:

Statics is the study of a body which is in stationary state.

$\Sigma \vec{F} = 0$

$\Sigma \vec{\Gamma} = 0$

$\Sigma F_x = 0$
$\Sigma F_y = 0$

The two requirements for a body to be in equilibrium:

1. The vector sum of all the external forces that act on the body must be zero.

2. The vector sum of all the external torques that act on the body measured about any possible point must be zero.

$\sum \tau = 0 \leftarrow$
$\tau = I \cdot \boxed{\alpha}$

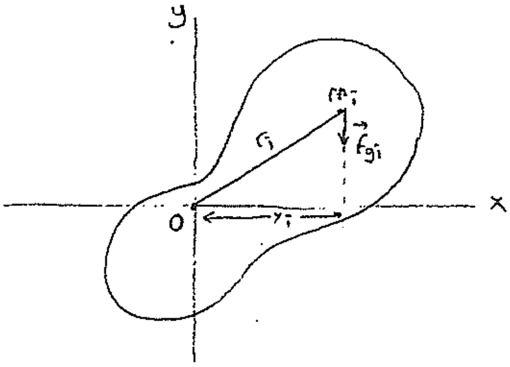
$\sum \vec{F}_{ext} = 0$

$\sum F_x = 0$ In a plane;
$\sum F_y = 0$ $\sum F_x = 0$
$\sum F_z = 0$ $\sum F_y = 0$

In torque equations:

$\sum \vec{\tau}_{ext} = 0$ In a plane;
$\sum \tau_z = 0$

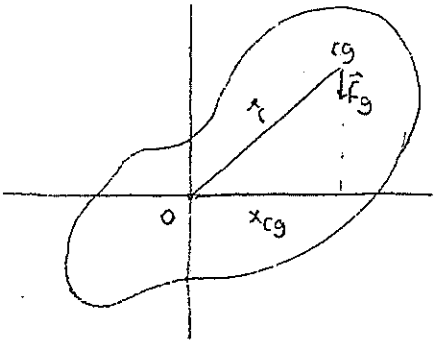
The center of Gravity:



$\tau_i = x_i \cdot F_{gi}$

$\sum \tau_{net} = \sum \tau_i = \sum x_i F_{gi}$

The gravitational force ($\vec{F}_g$) on a body effectively acts at a single point, called the center of gravity (cg) of the body.



$\tau = x_{cg} \cdot F_g \Rightarrow \tau = x_{cg} \cdot \sum F_{gi}$

$x_{cg} \sum_i F_{gi} = \sum_i x_i F_{gi}$

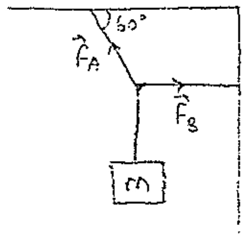
$F_{gi} = m g_i \rightarrow x_{cg} \sum_i m_i g_i = \sum_i x_i m_i g_i$

$$\boxed{x_{cg} \underbrace{\left(\sum_i m_i\right)}_{M = \sum_i m_i} = \sum_i x_i m_i}$$

$M = \sum_i m_i$ $x_{cg} = \frac{\sum_i m_i x_i}{M} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{M}$

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{\underbrace{m_1 + m_2 + \cdots}_{M}}$$

x:



If the system is in equilibrium, $F_A = ?$
$F_B = ?$

$m = 200$ kg
$g = 9.8$ m/s²

$\Sigma F_x = 0$
$\Sigma F_y = 0$

$F_{Ax} = -F_A \cdot \cos 60°$ $\quad mg = 200 \times 9.8 = 1960$ N $\quad F_B$ has only x component.

$F_{Ay} = F_A \cdot \sin 60°$

$\Sigma F_y = 0$

$F_A \sin 60° - mg = 0$

$F_A = 2260$ N

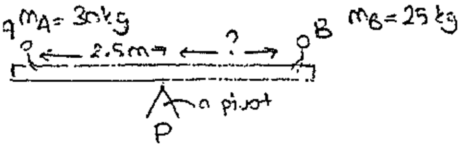
$\Sigma F_x = 0$

$F_B - F_A \cdot \cos 60° = 0$

$F_B = F_A \cos 60° = 2260 \cdot (\cos 60°)$

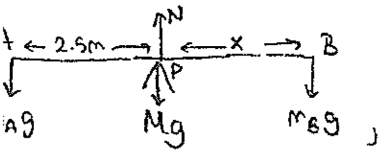
$F_B = 1130$ N

Ex:



A board of mass $M = 2.0$ kg serves as a seesaw for two children. A sits 2.5 m from the pivot point, P. At what distance x from the pivot must child B of mass 25 kg place herself to balance the seesaw. (Assume the board is uniform and centered over the pivot.)

Free body diagram



$\Sigma \Gamma = 0$

$m_A \cdot g \cdot (2.5) - m_B \cdot g \cdot x = 0$

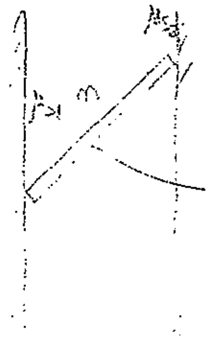
$\Sigma F_y = 0$

$N - Mg - m_A g - m_B g = 0$

$\Sigma \vec{F} = 0$

$\Sigma \vec{\Gamma} = 0$

Ex:



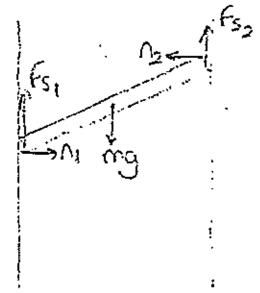
w=0.90 m

$\mu_{s_1} = 1.2$ $\mu_{s_2} = 0.80$ m = 0.60 kg

A thin uniform rod

Assume that the friction forces are both at a maximum and the rod is just to slide down.

a) Draw the free body diagram for the rod



b) Find the magnitude of the horizontal and vertical components of the forces exerted by each way on the rod.

$\Sigma \vec{F} = 0$

$\Sigma F_x = 0$

$\Sigma F_y = 0$

in x

$n_1 - n_2 = 0$

$n_1 = n_2$

in y

$f_{s_1} + f_{s_2} - mg = 0$

$f_{s_1} + f_{s_2} = mg \Rightarrow \mu_{s_1} n_1 + \mu_{s_2} n_2 = mg$

$f_{s_1} = \mu_{s_1} n_1$ $(\mu_{s_1} + \mu_{s_2}) n_1 = mg$

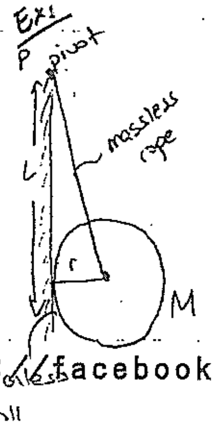
$f_{s_2} = \mu_{s_2} . n_2$ $n_1 = \dfrac{mg}{\mu_{s_1} + \mu_{s_2}}$

$n_1 = n_2 = \underline{\underline{3\,N}}$

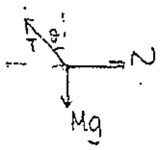
$f_{s_1} = \mu_{s_1} . n_1 = (1.2) . 3 = 3.6 \underline{\underline{N}}$

$f_{s_2} = \mu_{s_2} . n_2 = (0.8) . 3 = 2.4 \underline{\underline{N}}$

Ex:



Find the tension in the rope and the force on the sphere from the wall.



$\Sigma F_x = 0$ $\quad N - T\sin\theta = 0 \quad (1)$

$\Sigma F_y = 0.$ $\quad T\cos\theta - Mg = 0 \quad (2)$

$$T = \frac{Mg}{\cos\theta}$$

$$N = T\sin\theta$$

$$N = \frac{Mg}{\cos\theta} \cdot \sin\theta$$

$$T = \frac{w}{L/\sqrt{r^2+L^2}}$$

$$\boxed{N = \frac{wr}{L}}$$

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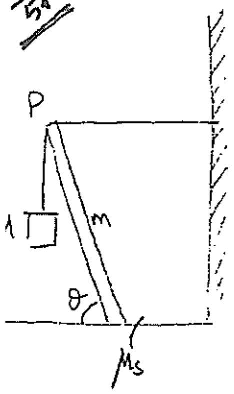


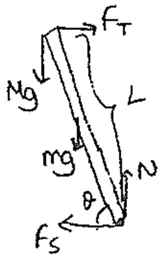
a) Find an expression for the maximum mass m

Equilibrium

$\Sigma F_x = 0$

$\Sigma F_y = 0$

free-body diagram:



$$f_T - f_s = 0 \quad (1) \longrightarrow T - \mu_s \cdot N = 0$$

$$N - Mg - mg = 0 \quad (2)$$

$$\Sigma \Pi = 0$$

$$Mg\,(L\cos\theta) - T(L\sin\theta) + mg\left(\frac{L}{2}\cdot\cos\theta\right) = 0 \quad (3)$$

$$M = \frac{m}{2} \cdot \left(\frac{2\mu_s \cdot \sin\theta - \cos\theta}{\cos\theta - \mu_s \cdot \sin\theta}\right)$$

$$\cos\theta - \mu_s \cdot \sin\theta > 0$$

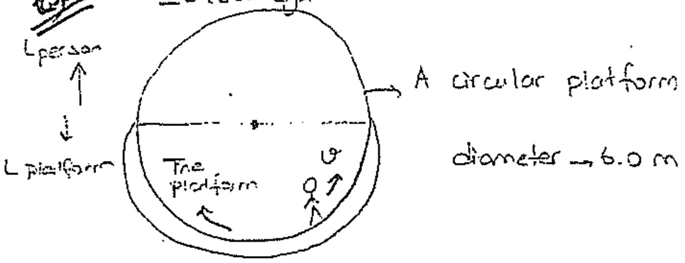
$$\mu_s < \cot\theta$$

b) find the magnitude of the reaction force at the floor.

$$R = \sqrt{N^2 + f_s^2}$$

$$R = \sqrt{N^2 + (\mu_s . N)^2}$$

Ex: $I = 1500 \ kg.m^2$



Initially it is at rest

$\vartheta = 4.2 \ m/s$ (with respect to ground.

Calculate the angular velocity of the platform.

$L_i = 0$

$$L_{person} = \overset{I}{\overbrace{(m.R^2)}} . \overset{\omega}{\overbrace{\frac{\vartheta}{R}}} = m.\vartheta.R$$

$$L_{platform} = - I.\omega$$

$$0 = m.\vartheta.R - I.\omega$$

$$\boxed{\omega = \frac{m.\vartheta.R}{I}}$$

CHAPTER 13

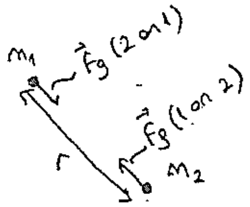
UNIVERSAL GRAVITATION

→ The variation of weight with altitude

→ The orbits of satellites around the earth

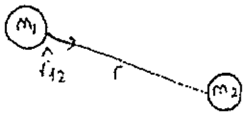
→ The orbits of planets around the sun

$$F_g = G . \frac{m_1 . m_2}{r^2} \quad \text{(Law of gravitation)}$$



$F_g (1 \text{ on } 2) = F_g (2 \text{ on } 1)$

G → Gravitational constant



$$\vec{F}_{12} = -G \cdot \frac{m_1 \cdot m_2}{r^2} \cdot \hat{r}_{12}$$

In SI units, $G = 6.6742 \times 10^{-11}\ N.m^2/kg^2$

The weight w of a small body of mass m at the earth's surface

$$w = F_g = \frac{G.m_E.m}{R_E^2} \qquad \boxed{g = \frac{G.m_E}{R_E^2}}$$ (Acceleration due to gravity at the earth's surface)

$w = mg$

$R_E = 6380\ km = 6,38 \times 10^6\ m$

$g = 9.8\ m/s^2$

$$m_E = \frac{gR_E^2}{G}$$

Gravitational Potential Energy:

$U = mgy$

$$F_g = \frac{G.m_E.m}{r^2}$$

The work w_{grav} done by gravitational force:

$$w_{grav} = \int_{r_1}^{r_2} F_r\, dr$$ $F_r \rightarrow$ the radial component of F_g

F_r is negative $\quad F_r = -\frac{G.m_E.m}{r^2}$

$\downarrow$

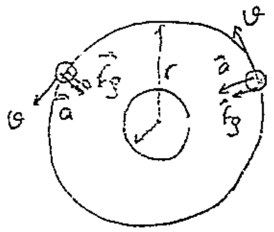
(F_g is directly toward the center of the earth.)

$$w_{grav} = -G.m_E.m \int_{r_1}^{r_2} \frac{dr}{r^2}$$

$$\boxed{w_{grav} = \frac{G.m_E.m}{r_2} - \frac{G.m_E.m}{r_1}}$$ $F_g \rightarrow$ conservative

$w_{grav} = -\Delta U = U_1 - U_2 \qquad \boxed{U = -\frac{G.m_E.m}{r}}$ (Gravitational potential energy)

$$W_{grav} = mg(r_1 - r_2)$$



$$a_{rad} = \frac{\vartheta^2}{r}$$

$$f_g = G\,\frac{m_E m}{r^2}$$

$r \rightarrow$ the radius of satellite

$$\frac{G m_E m}{r^2} = \frac{m\vartheta^2}{r}$$

$$\boxed{\vartheta = \sqrt{\frac{G.m_E}{r}}}$$ (Circular orbit)

$$\vartheta = \frac{2\pi r}{T}$$

$$T = 2\pi r\sqrt{\frac{r}{G.m_E}} = \frac{2.\pi.r^{3/2}}{\sqrt{G.m_E}}$$

$$E = K + U$$

$$K = \frac{1}{2}m\vartheta^2 \quad U = -G\,\frac{m_E.m}{r}$$

$$E = \frac{1}{2}m\left(\frac{G.m_E}{r}\right) - \frac{G.m_E.m}{r}$$

$$\boxed{E = -\frac{G.m_E.m}{2r}}$$ (Circular orbit)

→ The potential energy U at infinity is zero.

$$U = W = \frac{-G.M.m}{R}$$

Potential energy $U(r)$

Force function $f(r)$

$$F = \frac{-dU}{dr} = \frac{-d}{dr}\left(-\frac{G.M.m}{r}\right)$$

$$F = -\frac{G.M.m}{r^2}$$

→ $K = \frac{1}{2}m\vartheta^2$ (a projectile with mass m)
radius R

$$U = -\frac{G.M.m}{R}$$

At infinity $K=0$

$U=0$

$$K+U=\frac{1}{2}m\vartheta^2+\left(-\frac{G.M.m}{R}\right)=0 \Rightarrow \vartheta=\sqrt{\frac{2.G.M}{R}}$$ (escape speed)

Kepler's Law and the Motion of Planets:

1. The Law of Orbits

$a \rightarrow$ semimajor axis

$e \rightarrow$ eccentricity

$b \rightarrow$ semiminor axis

2.

$r\Delta\theta \rightarrow$ base

$r \rightarrow$ height

$$\Delta A \cong \left(\frac{1}{2}r\Delta\theta\right).r=\frac{1}{2}r^2\Delta\theta$$

$$\frac{dA}{dt}=\frac{1}{2}r^2\frac{d\theta}{dt}=\frac{1}{2}r^2\omega$$

Angular momentum is conserved.

$$L=rP_1=r(m\vartheta_1)+r(m\omega r)$$

$$=mr^2\omega$$

$$\frac{dA}{dt}=\frac{1}{2}r^2\omega \Rightarrow r^2\omega=\frac{2dA}{dt}$$

$$L=\left(\frac{2dA}{dt}\right)m \Rightarrow \frac{dA}{dt}=\frac{L}{2m}$$

3. The Law of Periods

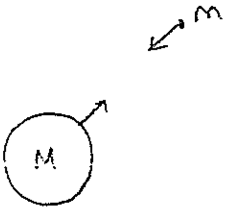
$$\frac{G.M.m}{r^2}=m.(\omega^2.r) \qquad \omega=\frac{2\pi}{T}$$

$$T^2=\left(\frac{4\pi^2}{GM}\right)r^3$$

$$\left(\frac{T_1}{T_2}\right)^2=\left(\frac{a_1}{a_2}\right)^3$$

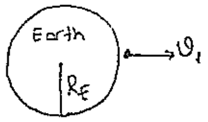
a = semi major axis

$$\frac{a_1^3}{T_1^2}=\frac{a_2^3}{T^2} \qquad \boxed{\frac{R^3}{T^2}}$$ should be the same for all planets



$$\vec{g} = \frac{\vec{F}_g}{m} = \frac{\frac{G.M.m}{R^2}}{m} = \underline{\underline{\frac{G.M}{R^2}}}$$

Ex: A 1000 kg rocket is fired straight away from the surface of the earth. What speed does the rocket need to "escape" from the gravitational pull of the earth and never return?



Assume a nonrotating earth.

$r_1 = R_E$

$r_2 = \infty$

$\vartheta_2 = 0$

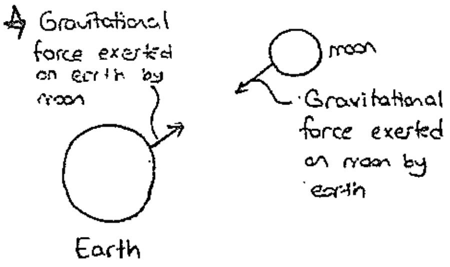
Use energy conservation:

$$K_2 + U_2 = K_1 + U_1$$

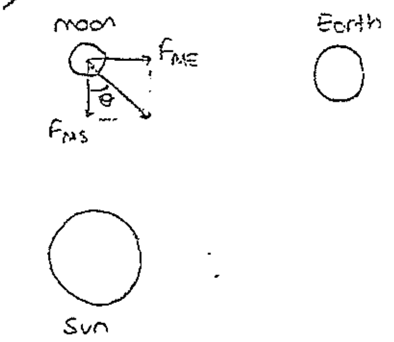
$$0 + 0 = \frac{1}{2} m \vartheta_1^2 - G.\frac{m_E.m}{r_1}$$

$$\vartheta_1^2 = \frac{2.G.m_E}{r_1} \Rightarrow \boxed{\vartheta_1 = \sqrt{\frac{2.G.m_E}{R_E}}} \qquad (r_1 = R_E)$$

does not depend on rocket's mass



Ex:



Find the net force on the Moon due to the gravitational attraction of both the Earth and the Sun, assuming they are at right angles to each other.

$m_m = 7.35 \times 10^{22}$ kg

$m_E = 5.98 \times 10^{24}$ kg

$m_S = 1.99 \times 10^{30}$ kg

$r_{ME} = 3.84 \times 10^{8}$ m

$r_{MS} = 1.5 \times 10^{11}$ m.

$G = 6.67 \times 10^{-11}\ N.m^2/kg^2$

$$F_{ME} = \frac{G.m_E.m_m}{r_{mE}^2} = 1.99 \times 10^{20}\ N$$

$$F_{MS} = \frac{G.m_s.m_m}{r_{ms}^2} = 4.34 \times 10^{20}\ N$$

$$F = \sqrt{F_{ME}^2 + F_{MS}^2} = 4.77 \times 10^{20}\ N //$$

$$\tan\theta = \frac{F_{mE}}{F_{ms}} \Rightarrow \theta = \tan^{-1}\left(\frac{F_{mE}}{F_{ms}}\right) = 24.6° //$$

→ G versus g

G	g
Universal gravitational constant.	It is acceleration due to gravity.
It is the same everywhere.	$g = 9.8\ m/s^2$ at the surface of the Earth.
	g will vary by location.

$$\boxed{g = G.\frac{m_E}{R_E^2}}$$

If an object is some distance h above the Earth's surface r becomes $R_E + h$

$$\boxed{g = G.\frac{m_E}{(R_E+h)^2}}$$

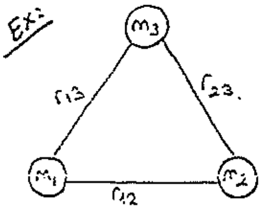
For any two particles the gravitational potential energy function becomes

$$\boxed{U = -G.\frac{m_1.m_2}{r}}$$

$F_g \propto 1/r^2$

$U \propto 1/r$

Ex:



three - particle system

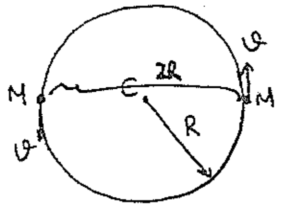
$$U_{total} = U_{12} + U_{13} + U_{23}$$

$$U_{total} = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right)$$

☆ U_{total} → The absolute value of U_{total} represents the work needed to separate the particles by an infinite distance.

Ex:

A binary star system consists of two stars, each of mass M, orbiting around their common center with radius R, as shown in figure.



a) Find the speed ϑ of each star in terms of G, M and R?

$\longrightarrow a_r \qquad a_r \longleftarrow$

$\longrightarrow F_r \qquad F_r \longleftarrow$

$$a_r = \frac{\vartheta^2}{R}$$

$$F_r = G.\frac{M.M}{(2R)^2}$$

$$F = m.a$$

$$G.\frac{M^{\cancel{2}}}{4R^{\cancel{2}}} = \cancel{M}.\frac{\vartheta^2}{\cancel{R}}$$

$$\vartheta^2 = \frac{G.M}{4R}$$

$$\vartheta = \sqrt{\frac{G.M}{4R}}$$

$$\vartheta = \frac{1}{2}\sqrt{\frac{G.M}{R}}$$ //

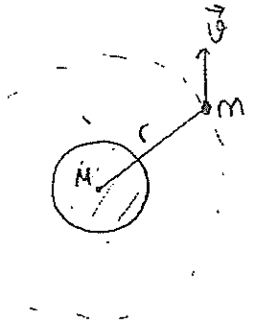
b) Find the period T of the stars in terms of G, M and R.

$$v = \frac{2\pi R}{T} \Rightarrow T = \frac{2\pi R}{v}$$

$$T = \frac{2.\pi.R}{\sqrt{\frac{G.M}{4R}}}$$

$$T = \frac{4\pi R^{3/2}}{\sqrt{GM}} \quad //$$

c) If their orbit radius is somehow doubled, now what would be the period T' in terms T you found in part B.



M >> m

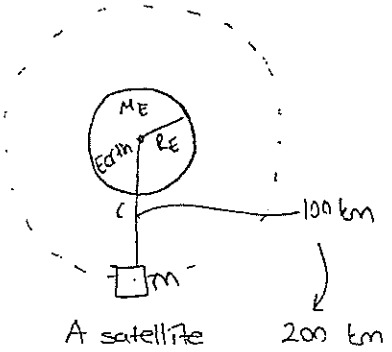
E=K+U

$$E = \frac{1}{2}mv^2 - \frac{G.M.m}{r} \qquad \frac{r}{2}\left(G.\frac{M.m}{r^2}\right) = m.\left(\frac{v^2}{r}\right)_{a_r} . \frac{r}{2}$$

$$\boxed{\frac{1}{2}.G.\frac{M.m}{r}} = \frac{1}{2}mv^2$$

$$E = \frac{1}{2}.G.\frac{M.m}{r} - G.\frac{M.m}{r} \Rightarrow \boxed{E = -\frac{G.M.m}{2r}}$$ the total energy

13/35 A satellite, m=1000 kg



How much energy must be added?

$$E_{tot} = -\frac{G.M.m}{2r}$$

$$\Delta E = E_{totf} - E_{tot\,i}$$

$$\Delta E = -\frac{G.M.m}{2r_f} - \left(-\frac{G.M.m}{2r_i}\right)$$

$$\Delta E = \frac{G.M.m}{2}\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

$G = 6.67 \times 10^{-11}\ N.m^2/kg^2$

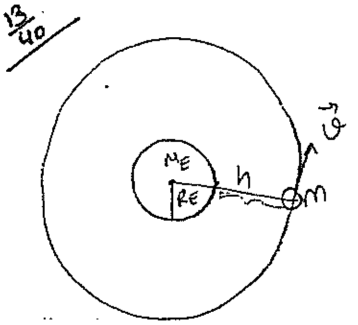
$M_E = 5.98 \times 10^{24}\ kg$

$R_E = 6370\ km$

$r_i = 6370 + 100\ km$ $\qquad$ $\Delta E = 4.65 \times 10^{8}\ J$

$r_f = 6370 + 200\ km$ $\qquad$ $\Delta E = 465\ MJ$ //

13/40



Altitude h

Assuming a circular orbit

How long does the satellite take to complete one orbit? $T = ?$

$$F = G.\frac{M_E.m}{R_E^2}$$

$$r = R_E + h$$

$$\frac{G.M_E.m}{(R_E+h)^2} = \frac{m.v^2}{(R_E+h)} \Rightarrow v = \sqrt{\frac{G.M_E}{(R_E+h)}}$$

$$2\pi(R_E+h) = v.T \Rightarrow T = \frac{2\pi(R_E+h)}{v}$$

$$T = 2\pi.\sqrt{\frac{(R_E+h)^3}{v}}$$ //

★The effect of the daily rotation $v_i = ?$

Min energy to satellite. $\Delta E_{min} = (\overset{\frac{1}{2}m\left(\frac{GM_E}{R_E+h}\right)^2}{K_f} + \overset{-G.\frac{M_E.m}{(R_E+h)}}{U_{gf}}) - (K_i + U_{gi})$

$$v_i = \frac{2\pi R_E}{\boxed{86400\,s}} \qquad U_{gi} = -\frac{G.M_E.m}{R_E}$$

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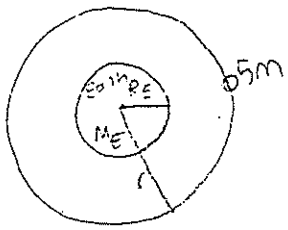


$$\vartheta_i = \left(\frac{G.M_E}{R}\right)^{1/2}$$

→ An explosion breaks the satellite into two parts:

m 4m

1 2

↓

Stationary

↓

falls

↓

Conservation of momentum:

$$(\Sigma m\vartheta)_i = (\Sigma m\vartheta)_f$$

$$5m\vartheta_i = m.0 + m.4\vartheta$$

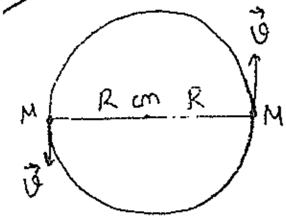
$$\vartheta = \frac{5}{4}.\vartheta_i$$

$$\vartheta = \frac{5}{4}.\left(\frac{G.M_E}{R}\right)^{1/2}$$

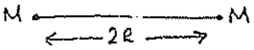
$$4m.R.\vartheta = 4m.r_f.\vartheta_f$$

$$\frac{1}{2}.4m.\vartheta^2 - \frac{G.M_E.4m}{r} = \frac{1}{2}.4m.\vartheta_f^2 - \frac{G.M_E.4m}{r_f}$$

EX:



Find the gravitational force of one star on the other star.



a) $F_g = \frac{G.M^2}{(2R)^2} = \frac{GM^2}{4R^2}$

b) Find the orbital speed of each star and the period of the orbit.

c) How much energy would be required to seperate the two stars to infinity?

$$E_i = E_f$$

$$K_i + U_i = \cancel{K_f} + \cancel{U_f}$$

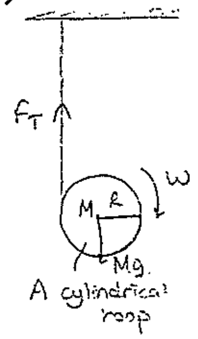
$K_i = 2.\frac{1}{2}.m.\vartheta^2$ $U_i = -\frac{G.M^2}{2R}$ $\boxed{\vartheta^2 = \frac{G.M}{4R}}$

$K_i + U_i + W_{other} = K_f + U_f$

$W_{other} = -(K_i + U_i)$

$W_{other} = \frac{G.M^2}{4R}$

Ex:



A cylindrical hoop

One end of the string is fixed and the hoop is allowed to fall vertically. Starting from rest, as the string unwinds.

Determine the angular acceleration and the angular momentum the hoop about its cm. $I = MR^2$

$\boxed{a_{cm} = R.\alpha}$

For the linear momentum of the cm

$\Sigma \vec{F} = M.a$

$\boxed{M.g - F_T = M.a}$

For angular motion about the cm

$\Sigma \Gamma = I.\alpha$

$\Sigma \Gamma = F_T.R$

$\boxed{F_T.R = I.\alpha}$

$F_T R = M.R^2.\alpha$

$\boxed{F_T = M.R.\alpha}$

$Mg - f_T = Ma$

$Mg - M.R.\alpha = M.\overset{\alpha.R}{a}$

$a_{cm} = \frac{g}{2}$

$\alpha = \frac{a_{cm}}{R} = \frac{g}{2R}$

$L = ?$

$\Sigma \Gamma = \frac{dL}{dt}$

$\frac{dL}{dt} = MR^2. \frac{g}{2R}$ $\frac{dL}{dt} = \frac{MRg}{2}$

$$\int dL = \int \frac{M.R.g}{2}\, dt$$

$$L = \frac{M.R.g}{2} \cdot t$$